

## Scaling approach to electron-electron interactions in a chaotic quantum dot

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A scaling theory is used to study the low-energy physics of electron-electron interactions in a double quantum dot. We show that the fact that electrons are delocalized over two quantum dots does not affect the instability criterion for the description of electron-electron interactions in terms of a “universal interaction Hamiltonian.”

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The statistical distribution of single-particle energy levels and wave functions in a chaotic quantum dot or disordered metal particle is described by random matrix theory.<sup>1,2</sup> The validity of random matrix theory as a statistical description of energy levels and wave functions follows from the existence of a large parameter, the dimensionless conductance  $g$  of the metal grain or the quantum dot.<sup>3,4</sup> (The dimensionless conductance is the ratio of the Thouless energy  $E_T$  and the mean level spacing  $\Delta$ .) The same large parameter  $g$  allows for a consistent and simple description of electron-electron interactions in quantum dots and metal grains, by means of the “universal interaction Hamiltonian,” which was proposed by Kurland, Aleiner, and Altshuler<sup>5</sup> (see also Ref. 6). According to Ref. 5, to leading order in  $g$ , the only relevant contributions to the interaction Hamiltonian are the capacitive charging energy, the long-range exchange interaction, and the “Cooper-channel” interaction, which is responsible for the superconducting instability.

The justification for the universal interaction Hamiltonian follows from the statistics of wave functions  $\phi_\alpha$  in disordered metal grains or chaotic quantum dots. Wave functions determine the matrix elements of the electron-electron interaction,

$$V_{\alpha\beta\gamma\delta} = \int d\mathbf{r}_1 d\mathbf{r}_2 \phi_\alpha(\mathbf{r}_1)^* \phi_\beta(\mathbf{r}_2)^* V(\mathbf{r}_1, \mathbf{r}_2) \phi_\gamma(\mathbf{r}_2) \phi_\delta(\mathbf{r}_1). \quad (1)$$

The absence of (long-range) wave function correlations in chaotic quantum dots causes interaction matrix elements to be self-averaging. Most averages are zero, except averages of “diagonal” interaction matrix elements  $V_{\alpha\beta\gamma\delta}$ , where the wave function indices coincide pairwise. Replacing interaction matrix elements  $V_{\alpha\beta\gamma\delta}$  by their ensemble average  $\langle V_{\alpha\beta\gamma\delta} \rangle$ , only the charging energy, exchange coupling, and Cooper channel interaction remain, thus leading to the universal interaction Hamiltonian. Small nonuniversal corrections to the interaction Hamiltonian follow from residual wave function correlations in disordered metal grains or quantum dots, which cause small fluctuations of the interaction matrix elements  $V_{\alpha\beta\gamma\delta}$  around their average. Typically, these fluctuations are a factor  $1/g$  smaller than the diagonal matrix elements.

Although the off-diagonal interaction matrix elements are a factor  $1/g$  smaller than the diagonal elements, they are many, and it is legitimate to ask what their role is. This

question was addressed by Murthy and co-workers using a renormalization-group approach in a series of papers.<sup>7-9</sup> These authors assumed Fermi-liquid interactions on time scales shorter than  $\hbar/E_T$ , and used random matrix theory to describe electron dynamics on time scales beyond  $\hbar/E_T$ . Successively integrating out states with highest energy, they found that the universal interaction Hamiltonian is stable for repulsive Fermi-liquid interactions and for weak attractive Fermi-liquid interactions, whereas an instability occurs when the attraction is sufficiently strong. Remarkably, Murthy *et al.* found that the critical attraction strength is a factor  $2\ln 2$  smaller than the attraction strength corresponding to the Pomeranchuk instability in the bulk Fermi liquid, thus creating a parameter regime where the bulk system is stable, whereas the finite-sized system is not.<sup>7-9</sup>

A renormalization-group treatment of interactions in chaotic quantum dots requires knowledge of how (nonuniversal) wave function correlations depend on the energy difference between the wave functions involved. The answer to this question depends on the detailed shape of the quantum dot and is different for diffusive<sup>10</sup> and ballistic<sup>11</sup> electron dynamics (see also Ref. 6). Murthy *et al.* bypass this problem by using the eigenfunction correlations of a  $g$ -dimensional random matrix for all wave functions with energy within  $E_T/2$  from the Fermi level, treating wave functions at larger energies as plane waves. Whereas the use of random matrix theory is justified for energies far below  $E_T$  only, it cannot be used to describe nonuniversal wave function statistics near the Thouless energy. Similarly, residual wave function correlations will persist for energies above  $E_T$ , which are not accounted for in Refs. 7-9. A correct treatment of wave function correlations around  $E_T$  is important for the renormalization group approach, since most of the renormalization of the interaction parameters takes place around that energy.

In this Rapid Communication we apply the renormalization-group scheme to the special case of a “double quantum dot,” see Fig. 2, inset. The double quantum dot consists of two quantum dots of roughly equal size coupled via a point contact with dimensionless conductance  $g/2 \gg 1$ . The Thouless energy  $E_T$  of the double-dot system is equal to  $g\Delta$ , where  $\Delta$  is the double-dot level spacing (which is half the single-dot level spacing). The dimensionless conductances of the two individual quantum dots are assumed to be much larger than  $g$ , so that random matrix theory and the universal interaction Hamiltonian can be used to describe

wave functions and interactions in each of the dots separately. The advantage of the double-dot geometry is that wave function correlations for energy differences near  $E_T$  can be calculated in detail, so that no approximations need to be made upon constructing the renormalization group for the electron-electron interactions. The analogy between the double-dot system studied here and the single quantum dot studied by Murthy *et al.* is that in the double-dot electrons are confined to one dot for times well below  $\hbar/E_T$ , but not for larger times, whereas in the ballistic quantum dot studied in Refs. 7–9 they have a well-defined momentum for times below  $\hbar/E_T$ , but not for longer times.

Our main finding, to be elaborated below, is that, once the correct nonuniversal wave function correlations near  $E_T$  are taken into account, the instability of the universal Hamiltonian occurs at precisely the same interaction strength as the instability of the double-dot system without point contact between the dots. Although this conclusion is reached for one specific geometry only, the structure of our calculation leads us to expect that the same is true for the more general Pomeranchuk-type instabilities studied by Murthy *et al.* In other words, we expect that the fact that Refs. 7–9 find a parameter regime where the bulk Fermi liquid is stable whereas the finite-sized system is not is an artifact of the use of random matrix theory to describe wave function statistics up to a distance  $E_T/2$  from the Fermi level.

We now describe the details of our calculation. For technical convenience, we consider a double quantum dot with spinless electrons and with broken time-reversal symmetry. Using random matrix theory to describe each of the dots separately, the noninteracting part of the Hamiltonian for the double quantum dot reads<sup>12</sup>

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix} + \sqrt{\frac{g}{8N}} H_{12}, \quad (2)$$

where  $H_1$  and  $H_2$  are  $N \times N$  Hermitian matrices modeling the Hamiltonians of the quantum dots without point contact and  $H_{12}$  is a  $2N \times 2N$  Hermitian matrix modeling the point contact connecting the two quantum dots. The elements of  $H_1$ ,  $H_2$ , and  $H_{12}$  are complex numbers taken from independent and identical Gaussian distributions. The rows and columns of  $H$  are labeled by roman numbers  $k=1, \dots, 2N$ , where  $k=1, \dots, N$  and  $k=N+1, \dots, 2N$  correspond to the left and right dots, respectively. Eigenvalues of  $H$  are denoted  $\varepsilon_\alpha$ , the corresponding eigenvector being written  $\phi_\alpha(k)$ . The size  $N$  of the random matrices  $H_1$  and  $H_2$  is of the order of the dimensionless conductance of the individual quantum dots and is taken to infinity at the end of the calculation.

The interaction Hamiltonian has the form<sup>13</sup>

$$H_{\text{int}} = \frac{1}{2} U_0 (\hat{n}_1 + \hat{n}_2)^2 + \frac{1}{2} U_1 (\hat{n}_1 - \hat{n}_2)^2, \quad (3)$$

where  $\hat{n}_1$  and  $\hat{n}_2$  are operators for the number of electrons in the two individual quantum dots. The first term in Eq. (3) corresponds to a “charging energy” for the double dot system, whereas the second term in Eq. (3) represents a dipolar

interaction. Upon changing to the basis of eigenstates of the noninteracting Hamiltonian  $H$ , the total Hamiltonian reads

$$\mathcal{H} = \sum_\alpha \varepsilon_\alpha \hat{\psi}_\alpha^\dagger \hat{\psi}_\alpha + \frac{1}{2} \sum_{\alpha\beta\gamma\delta} V_{\alpha\beta\gamma\delta} \hat{\psi}_\alpha^\dagger \hat{\psi}_\beta^\dagger \hat{\psi}_\gamma \hat{\psi}_\delta, \quad (4)$$

where  $\hat{\psi}_\alpha^\dagger$  and  $\hat{\psi}_\alpha$  are creation and annihilation operators for an electron in eigenstate  $\alpha$  of the noninteracting Hamiltonian  $H$  and

$$V_{\alpha\beta\gamma\delta} = \sum_{k,l} (U_0 + U_1 \sigma_k \sigma_l) \phi_\alpha^*(k) \phi_\beta^*(l) \phi_\gamma(l) \phi_\delta(k), \quad (5)$$

where  $\sigma_k = 1$  for  $k=1, \dots, N$  and  $\sigma_k = -1$  for  $k=N+1, \dots, 2N$ .

For large  $g$ , wave function elements  $\phi_\alpha(k)$  are independently distributed Gaussian random numbers,

$$\langle \phi_\alpha(k) \phi_\beta(l) \rangle = \frac{1}{2N} \delta_{kl} \delta_{\alpha\beta}, \quad (6)$$

corrections to Eq. (6) being of order  $1/g$ . As a result, interaction matrix elements are self-averaging; fluctuations are of relative order  $1/g$ . Only diagonal elements have a nonzero average,

$$\langle V_{\alpha\beta\gamma\delta} \rangle = U_0 \delta_{\alpha\delta} \delta_{\beta\gamma}. \quad (7)$$

Replacing the interaction matrix elements by their average, we find that interactions are described by the reduced interaction Hamiltonian

$$H_{\text{int}} = \frac{U_0}{2} \hat{n}^2, \quad (8)$$

where  $\hat{n} = \hat{n}_1 + \hat{n}_2$  is the total number of electrons in the double quantum dot. Note that, in comparison to Eq. (3), the dipolar interaction has disappeared because the electron wave functions are delocalized over of the entire double-dot system. Equation (8) is the equivalent of the universal interaction Hamiltonian for the double quantum dot; the disappearance of the dipolar interaction is the double-dot counterpart of the disappearance of all nonzero-mode Fermi-liquid interactions in the original construction of the universal interaction Hamiltonian.<sup>5,6</sup>

In order to study the importance of the many residual interaction matrix elements that are of order  $1/g$ , we perform a renormalization-group analysis, following Refs. 7–9 (see also Ref. 14). This analysis is reminiscent of Anderson’s “poor man’s” treatment of the Kondo problem.<sup>15</sup> In order to find the effective interaction for electrons at the Fermi level  $\varepsilon_F$ , one successively integrates out states at energies far away from  $\varepsilon_F$ . Writing the cutoff energy as  $M\Delta/2$ , we calculate the change of the effective interaction parameters  $\bar{U}_0$  and  $\bar{U}_1$  upon changing  $M$  to  $M' < M$  within second-order perturbation theory, see Fig. 1

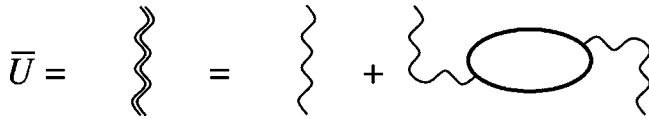


FIG. 1. Diagrammatic representation of the effective interaction  $\bar{U}$  to second order in perturbation theory.

$$\bar{U}_0(M') - \bar{U}_0(M) = \bar{U}_0^2 \sum_{k,l} \sum_{\mu\nu}' \frac{n_F(\varepsilon_\mu) - n_F(\varepsilon_\nu)}{\varepsilon_\mu - \varepsilon_\nu} \times \phi_\mu^*(k) \phi_\mu(l) \phi_\nu^*(l) \phi_\nu(k), \quad (9)$$

$$\bar{U}_1(M') - \bar{U}_1(M) = \bar{U}_1^2 \sum_{k,l} \sigma_k \sigma_l \sum_{\mu\nu}' \frac{n_F(\varepsilon_\mu) - n_F(\varepsilon_\nu)}{\varepsilon_\mu - \varepsilon_\nu} \times \phi_\mu^*(k) \phi_\mu(l) \phi_\nu^*(l) \phi_\nu(k). \quad (10)$$

where  $n_F(\varepsilon)$  is the Fermi function and the sum over intermediate states is such that only states  $\mu$  and  $\nu$  with at least one of the energies  $\varepsilon_\mu$  or  $\varepsilon_\nu$  in the cutoff region  $M'\Delta/2 < |\varepsilon - \varepsilon_F| < M\Delta/2$  are to be included. We omitted exchange contributions to the effective interaction, which are unimportant for the large- $g$  limit we consider here. We follow Refs. 7–9 in replacing the product of the eigenfunctions  $\phi_\mu$  and  $\phi_\nu$  of the intermediate energies  $\varepsilon_\mu$  and  $\varepsilon_\nu$  in Eqs. (9) and (10) by its ensemble average. However, we deviate from Refs. 7–9 in keeping the precise dependence of the ensemble average of the wave functions  $\phi_\mu$  and  $\phi_\nu$  on the energy difference  $\varepsilon_\mu - \varepsilon_\nu$ . Repeating the analysis of Ref. 16 for the Hamiltonian (2), the relevant wave function average is found to be (see also Ref. 17)

$$\langle \phi_\nu^*(k) \phi_\mu(k) \phi_\mu^*(l) \phi_\nu(l) \rangle = \frac{g \sigma_k \sigma_l \Delta^2}{4N^2 [g^2 \Delta^2 + \pi^2 (\varepsilon_\mu - \varepsilon_\nu)^2]} + \frac{\delta_{\mu\nu} + \delta_{kl}}{4N^2} - \frac{1 + \sigma_k \sigma_l}{8N^3}. \quad (11)$$

Substituting this into Eqs. (9) and (10), we find that only the nonuniversal first term on the right-hand side of Eq. (11) contributes in the limit  $N \rightarrow \infty$ . Changing to dimensionless interaction parameters  $\bar{u}_j = \bar{U}_j / \Delta$ ,  $j=0,1$ , and replacing the difference equations (9) and (10) by a differential equation, one thus finds

$$\frac{d\bar{u}_0}{dM} = 0, \quad (12)$$

$$\frac{d\bar{u}_1}{dM} = \frac{\bar{u}_1^2}{g} \ln \left( \frac{4g^2 + M^2 \pi^2}{g^2 + M^2 \pi^2} \right). \quad (13)$$

The flow equations (12) and (13) are solved with the boundary conditions  $\bar{u}_j \rightarrow u_j = U_j / \Delta$ ,  $j=0,1$ , if  $M \rightarrow \infty$ . Integrating Eqs. (12) and (13), one finds that  $u_0$  does not flow,  $\bar{u}_0(M) = u_0$  for all  $M$ , whereas

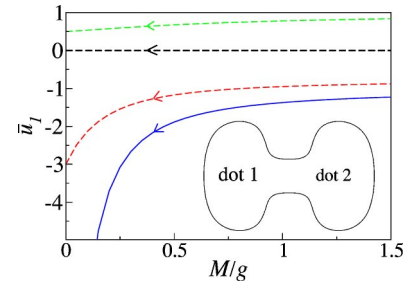


FIG. 2. (Color online) Renormalization-group flow of the effective dipolar coupling  $\bar{u}_1$  as a function of the cutoff  $M$ . The solid curve corresponds to the critical dipolar interaction strength  $u_1 = -1$ . The other curves are (from bottom to top) for  $u_1 = -0.75$ , 0, and 1. Inset: schematic drawing of the double quantum dot.

$$\bar{u}_1(M) = \left[ \frac{1}{u_1} + 1 - \frac{4}{\pi} \arctan \frac{M\pi}{2g} + \frac{2}{\pi} \arctan \frac{M\pi}{g} - \frac{M}{g} \ln \left( \frac{4g^2 + M^2 \pi^2}{g^2 + M^2 \pi^2} \right) \right]^{-1}. \quad (14)$$

In Fig. 2, the solution of the flow equation for the effective dipolar interaction  $\bar{u}_1$  is shown for various values of the unrenormalized interaction  $u_1$ .

In order to address the stability of the universal interaction Hamiltonian, we analyze Eq. (14) in the limit  $M \downarrow 0$ ,

$$\bar{u}_1(M) = \left( \frac{1}{u_1} + 1 - \frac{2M}{g} \ln 2 \right)^{-1} \quad \text{for } M \downarrow 0. \quad (15)$$

For  $u_1 > -1$ , the effective interaction strength  $\bar{u}_1$  remains bounded as  $M \downarrow 0$ . This implies that the corresponding interaction matrix elements at the Fermi level remain of order  $1/g$ , justifying the use of the universal interaction Hamiltonian for those values of the dipolar interaction. It is only for the critical dipolar attraction strength  $u_1 = -1$  that  $\bar{u}_1$  diverges upon taking the cutoff energy  $M\Delta$  to zero. This is precisely at the same interaction strength as the location of the instability in the absence of interdot tunneling.

The renormalization approach of Murthy *et al.* differs from ours in two respects. First, in Refs. 7–9 there is no flow of the interaction parameters for  $M > g$ . Second, in order to describe the flow for  $M < g$ , Murthy *et al.* replace the eigenfunction average (11) by the average of eigenfunctions of a random matrix of size  $g$ ,

$$\langle \phi_\nu^*(k) \phi_\mu(k) \phi_\mu^*(l) \phi_\nu(l) \rangle = \frac{\delta_{\mu\nu} + \delta_{kl}}{g^2} - \frac{1}{g^3}. \quad (16)$$

One then obtains the following flow equations for the effective dipolar interaction strength  $\bar{u}_1$ :

$$\frac{d\bar{u}_1}{dM} = 0 \quad \text{if } M > g, \quad (17a)$$

$$\frac{d\bar{u}_1}{dM} = \frac{2\bar{u}_1^2 \ln 2}{g} \quad \text{if } M < g. \quad (17b)$$

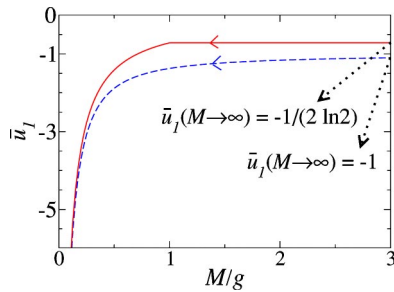


FIG. 3. (Color online) Comparison of exact renormalization group flow (dashed) and the flow according to the calculation scheme of Refs. 7–9 (solid). Flows are shown for the critical value of the dipolar interaction strength  $u_1$ .

The flow equations (17) agree with the exact flow equations for the double-dot system only for  $M \ll g$  and  $M \gg g$ , but not for the intermediate range  $M \sim g$ . The solution of the erroneous flow equations (17) is

$$\bar{u}_1(M) = \begin{cases} u_1 & \text{if } M > g, \\ [u_1^{-1} + 2(1 - M/g)\ln 2]^{-1} & \text{if } M < g. \end{cases} \quad (18)$$

One verifies that in this calculation scheme, the universal interaction Hamiltonian is stable for  $u_1 > -1/2 \ln 2$  only, so that there is a range of dipolar interaction strengths  $-1 < u_1 < -1/2 \ln 2$  for which the separate dots are stable against the formation of a dipolar charge distribution, whereas the coupled dots are not. Our exact calculation shows that such a result is incorrect. In Fig. 3 we compare the flow of Eq. (18) and the exact flow of Eq. (14) for the critical value of  $u_1$ . Although it is only in the range  $M \sim g$  that the renormaliza-

tion group flow of Refs. 7–9 and the exact flow for the double-dot differ, the flow in the range  $M \sim g$  is crucial in determining the value of the interactions at which the universal interaction Hamiltonian becomes unstable.

Before concluding, we would like to make three remarks about the renormalization-group calculation presented here. First, the one-loop renormalization-group result (17) is exact in the large- $g$  limit. This follows from the same arguments as used to establish the validity of one-loop renormalization group in the work of Murthy *et al.*<sup>9</sup> Second, in the exact calculation performed here all flow of interaction parameters arises from the first, nonuniversal term in the wave function correlator (11), which is off diagonal in the wave function indices. This is opposite to the calculation of Refs. 7–9, where the flow arises from a universal and diagonal wave function correlator. Third, mathematically, the fact that the instability of the universal interaction Hamiltonian occurs precisely at  $u_1 = -1$  is a consequence of the Lorentzian energy dependence of the first, nonuniversal term in the wave function correlator (11). A Lorentzian is generic for nonuniversal wave function correlations in both diffusive and ballistic quantum dots<sup>10,11</sup> for which  $\sigma_k$  and  $g$  in Eq. (11) are replaced by eigenfunctions and eigenvalues of the diffusion operator or the Perron-Frobenius operator, respectively, see, e.g., Ref. 6. It is because of this similarity that we believe that our calculational scheme, including our result for the critical interaction strength, extends to the general case.

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