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General strategies for discrimination of quantum states

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Abstract

We derive the general discrimination of quantum states chosen from a certain set, given an initial M copies of each state, and obtain the matrix inequality which describes the bound between the maximum probability of correctly determining and that of error. The former works are special cases of our results. © 1999 Elsevier Science B.V. All rights reserved.

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It is determined by the principle of quantum mechanics [1] that we can't perfectly discriminate or clone an arbitrary, unknown quantum state. But this important result doesn't prohibit discrimination and cloning strategies which have a limited success. Numerous projects have been made to this subject by many authors. Discrimination of states has a close connection with quantum measurement. The general approach of the quantum measurement theory considers a quantum system whose unknown state belongs to a finite, known set and devises the measurement which yields the most information about the initial state, where the figure of merit is the probability of a correct result or the mutual information [2]. We have known that a set of orthogonal quantum states can be discriminated perfectly. Approximate state determination of a set of non-orthogonal states

is also possible. Helstrom has found the absolute maximum probability of discriminating between two states, which is instead given by the well-known Helstrom limit [2]. Non-orthogonal quantum states can also, with some probability, be discriminated without approximation. Ivanovic [3], Dieks [4] and Peres [5] have showed that it is possible to discriminate exactly, which means with zero error probability, between a pair of non-orthogonal states and they derive the maximum probability of success called the IDP limit. The IDP limit is not the absolute maximum of the discrimination probability, but is rather the maximum subject to the constraint that the measurement never gives error results. Chelfes and Barnett [6–10] extend their results to the constraint of M initial copies of states and they also considered the results for n states. They found that n non-orthogonal states can be probabilistic discriminated without error if and only if they are linearly independent. Their works connect the approximate and exact discrimination and give an inequality that describes the

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relation between the probability of correctly and error discriminating. All their works have used an assumption that the probabilities of correct or error for discrimination are equal for different states. More recently, Duan and Guo [11] have found the maximum probabilities of exact discrimination of n linearly independent quantum states and for different states the maximum probability may be different. Exact discrimination attempts may give inconclusive results, but we can always know with certainty whether or not the discrimination has been a success. Recently, however, Massar and Popescu [12] and Derka et al. [13] have considered the problem of estimating a completely unknown quantum state, given M independent realizations.

In quantum mechanics, a combination of unitary evolution together with measurements often yields interesting results, such as the quantum programming [14], the purification of entanglement [15], and the teleportation [16] and preparation [17] of quantum states. Duan and Guo [11] used such a combination in the field of quantum cloning and showed that the states secretly chosen from a certain set $S = \{|\Psi_1\rangle, |\Psi_2\rangle, \dots, |\Psi_n\rangle\}$ can be probabilistically cloned if and only if $|\Psi_1\rangle, |\Psi_2\rangle, \dots, |\Psi_n\rangle$ are linearly independent. They derive the best cloning efficiencies and also extend their results to the discrimination of a set of states and give the optimal efficiencies for exactly probabilistically discriminating. In this paper we'll also use such a combination and construct a general discrimination of the states secretly chosen from a certain set S , given M initial copies of each state. We derive the matrix inequality which describes the bound between the maximum probability of correctly determining and that of error. We give the most general results about quantum discrimination in a finite set of states and prove that the former works can be derived from our results using different conditions.

Consider a set of quantum states $S = \{|\Psi_1\rangle, |\Psi_2\rangle, \dots, |\Psi_n\rangle\}$, $S \subset \mathcal{H}$, where \mathcal{H} is an n -dimensional Hilbert space. If there are M quantum systems, all of which are prepared in the same states, then the possible states of the combined system are the M -fold tensor products

$$|\Psi_i\rangle^{\otimes M} = |\Psi_i\rangle_1 \otimes \dots \otimes |\Psi_i\rangle_M. \quad (1)$$

Any operation in quantum mechanics can be represented by a unitary evolution together with a measurement. We use a unitary evolution \hat{U} and yield

$$\begin{aligned} \hat{U}|\Psi_i\rangle_A^{\otimes M}|P_0\rangle = & \sqrt{\gamma_i}|\varphi_i\rangle_A|P_0\rangle + \sum_j t_{ij}|\varphi_j\rangle_A|P_0\rangle \\ & + \sum_j c_{ij}|\varphi_j\rangle_A|P_1\rangle, \end{aligned} \quad (2)$$

where $t_{ii} = 0$, $\gamma_i \geq 0$, $t_{ij} \geq 0$, $\{|\varphi_i\rangle_A, i = 1, 2, \dots, n\}$ are a set of orthogonal states in Hilbert space $\mathcal{H}^{\otimes M}$, $\{|P_0\rangle, |P_1\rangle\}$ are the orthogonal basis states of the probe system P , and A represents the system of initial states. After the unitary evolution we measure the probe system P . If we get $|P_1\rangle$, the discrimination fails and we discard the output, so we call this situation an inconclusive result, else if we get $|P_0\rangle$, we then measure system A and if we get $|\varphi_k\rangle_A$, we determine the initial states as $|\Psi_k\rangle^{\otimes M}$. We can see the discrimination may be correctly success (when $k = i$, the probability is $P_D^{(i)} = \gamma_i$) or have errors (when $k \neq i$, the all probability is $P_E^{(i)} = \sum_k t_{ik}^2$) when the initial state is $|\Psi_i\rangle^{\otimes M}$. We can also give the inconclusive probability $P_I^{(i)} = 1 - P_D^{(i)} - P_E^{(i)}$.

Unitary evolution \hat{U} exist if and only if ¹

$$X^{(M)} = (\sqrt{\Gamma} + T)(\sqrt{\Gamma} + T^+) + CC^+, \quad (3)$$

where the $n \times n$ matrices $X^{(M)} = [\langle \Psi_i | \Psi_j \rangle^M]$, $\Gamma = \text{diag}(\gamma_1, \gamma_2, \dots, \gamma_n)$, $C = [c_{ij}]$, $T = [t_{ij}]$, we call Γ the correctly discriminating probabilistic matrix and T the error discriminating probabilistic matrix. CC^+ is semi-positive definite, that yields

$$X^{(M)} - (\sqrt{\Gamma} + T)(\sqrt{\Gamma} + T^+) \geq 0, \quad (4)$$

where ≥ 0 means semi-positive definite. This inequality gives a general bound among the initial states matrix X , correctly discriminating the probabilistic matrix Γ and error discriminating probabilistic matrix T . We don't make constraints on the initial states, which means we don't demand that the initial states must be linearly independent. In the following we discuss inequality (4) and find that many former works about discrimination can be derived from this inequality.

¹ See Ref. [11], Duan and Guo give a Lemma that if two sets of states $|\phi_1\rangle, |\phi_2\rangle, \dots, |\phi_n\rangle$, and $|\tilde{\phi}_1\rangle, |\tilde{\phi}_2\rangle, \dots, |\tilde{\phi}_n\rangle$ satisfy the condition $\langle \phi_i | \phi_j \rangle = \langle \tilde{\phi}_i | \tilde{\phi}_j \rangle$, there exists a unitary operate U to make $U|\phi_i\rangle = |\tilde{\phi}_i\rangle$, ($i = 1, 2, \dots, n$).

Denote $B = \sqrt{\Gamma} + T$, we rewrite inequality (4) as $X^{(M)} - BB^+ \geq 0$. (5)

We begin our discussion by giving a condition that the yields $\{|\Psi_i\rangle^{\otimes M}, i=1,2,\dots,n\}$ are linearly independent, that is

$$\sqrt{\gamma_i} > \sum_j t_{ij}, \quad (6)$$

where $i=1,2,\dots,n$. Condition (6) yield $X^{(M)}$ is a positive definite matrix², which means that $\{|\Psi_i\rangle^{\otimes M}, i=1,2,\dots,n\}$ are linearly independent. We can derive from inequality (6) that $P_D^{(i)} > P_E^{(i)}$, which means the correctly discriminating probability is greater than that of error. We consider the special situation that $T=0$. To obtain the maximum of correctly discriminating probability $\gamma_i > 0$, inequality (6) must be satisfied and $\{|\Psi_i\rangle^{\otimes M}, i=1,2,\dots,n\}$ must be linearly independent, which means that only linearly independent states can be discriminated with non-zero probability if we demand that there is no error existence. Such a result has been obtained by Duan and Guo [11] and Chefles and Barnett [6]. We can also give the maximum discriminating probability that is determined by such inequality

$$X^{(M)} - \Gamma \geq 0. \quad (7)$$

This inequality is just a generalization of the optimal efficiencies for exactly probabilistically discriminating which has been given by Duan and Guo [11] and we also obtain such inequality in [18].

In the following discussion we are only concerned with the discrimination of two states. This situation is the most important and many valuable works have treated this. We will give the most general bound between the correctly discriminating probability and that of error in this situation using inequality (4) and this bound comes to the former results in different special conditions.

Consider a quantum system prepared in one of the two states $|\psi_{\pm}\rangle^{\otimes M}$. We are not told which of the

states the system is in, although we do know that it has some probability of being in either. Denote $P_{\text{IP}} = \langle \psi_+ | \psi_- \rangle^M$. We represent X , Γ and T as

$$X = \begin{pmatrix} 1 & P_{\text{IP}} \\ P_{\text{IP}}^* & 1 \end{pmatrix}, \quad \Gamma = \text{diag}(\sqrt{P_D^+}, \sqrt{P_D^-}),$$

$$T = \begin{pmatrix} 0 & \sqrt{P_E^+} \\ \sqrt{P_E^-} & 0 \end{pmatrix}$$

and inequality (4) yields

$$(1 - P_D^+ - P_E^+)(1 - P_D^- - P_E^-) \geq (P_{\text{IP}} - \sqrt{P_D^+ P_E^-} - \sqrt{P_D^- P_E^+}) \times (P_{\text{IP}}^* - \sqrt{P_D^+ P_E^-} - \sqrt{P_D^- P_E^+}). \quad (8)$$

For $P_I^+ = 1 - P_D^+ - P_E^+$, $P_I^- = 1 - P_D^- - P_E^-$, we can rewrite inequality (8) as

$$P_I^+ P_I^- \geq |P_{\text{IP}} - \sqrt{P_D^+ P_E^-} - \sqrt{P_D^- P_E^+}|^2. \quad (9)$$

This inequality is just the most general bound among the discriminating probabilities of correct (P_D^+ , P_D^-), error (P_E^+ , P_E^-) and inconclusive (P_I^+ , P_I^-). In the following we will give some special conditions and simplify the inequality (9).

1. We let $P_I^+ = P_I^- = 0$, which means that we don't give inconclusive results, then inequality (9) yields

$$P_{\text{IP}} = \sqrt{P_D^+(1 - P_D^-)} + \sqrt{P_D^-(1 - P_D^+)}. \quad (10)$$

Eq. (10) gives a bound on the maximum correctly discriminating probabilities P_D^+ , P_D^- of the two states $|\psi_+\rangle$ and $|\psi_-\rangle$ given M initial copies of each states. We find P_{IP} must be real, which means if the inter-produce of two states $|\psi_+\rangle$ and $|\psi_-\rangle$ is not real we can't execute discrimination without inconclusive results. Furthermore we can suppose $P_D^+ = P_D^- = P_D$, and derive

$$P_D = \frac{1}{2} \left(1 + \sqrt{1 - P_{\text{IP}}^2} \right). \quad (11)$$

When $M=1$, Eq. (11) just gives the well-known Helstrom limit [2]

$$P_D = P_H = \frac{1}{2} \left(1 + \sqrt{1 - |\langle \psi_+ | \psi_- \rangle|^2} \right). \quad (12)$$

² Matrix B satisfy condition (6). We can prove that the determinant of such matrix is non-zero, which means BB^+ is positive definite. So $X^{(M)}$ must be positive definite to satisfy inequality (5).

Eq. (11) gives the absolute maximum probability of discriminating between two states $|\psi_{\pm}\rangle$ with given M initial copies. The measurement it represents does not give inconclusive results, but will incorrectly identify the states with probability $1 - P_D$.

2. We suppose $P_{IP} = P_{IP}^*$, $P_D^+ = P_D^- = P_D$, $P_E^+ = P_E^- = P_E$, so $P_I^+ = P_I^- = P_I$ and Eq. (9) yields

$$\frac{1}{2}(P_{IP} - P_I) \leq \sqrt{P_E P_D} \leq \frac{1}{2}(P_{IP} + P_I). \quad (13)$$

Inequality (13) gives a general lower bound on the combination of errors and inconclusive result and corresponds to a family of measurements which optimally interpolates between the Helstrom and IDP limits. If $P_{IP} \geq P_I$, we can get

$$\frac{1}{4}(P_{IP} - P_I)^2 \leq P_E P_D \leq \frac{1}{4}(P_{IP} + P_I)^2. \quad (14)$$

When $M = 1$, the left-hand side of inequality (14) is just the inequality that has been obtained by Chefles and Barnett [8]. If $P_{IP} < P_I$, we have

$$P_E P_D \leq \frac{1}{4}(P_{IP} + P_I)^2. \quad (15)$$

3. We suppose $P_E^+ = P_E^- = 0$, and inequality (8) yields

$$(1 - P_D^+)(1 - P_D^-) \geq |P_{IP}|^2. \quad (16)$$

So we obtain

$$\frac{P_D^+ + P_D^-}{2} \leq 1 - |\langle \psi_+ | \psi_- \rangle|^M. \quad (17)$$

When $P_D^+ = P_D^- = P_{IDP}$, we give $P_{IDP} = 1 - |\langle \psi_+ | \psi_- \rangle|^M = 1 - P_{IP}$, which is a generalization of the IDP limit.

Denote $P_S = P_D + P_E$. It is obvious that P_S is the probability with which we measure the probe system and get $|P_0\rangle$ after the unitary evolution in Eq. (2). Comparing with [9], we can say that P_S is just the probability that two states can be separated by an arbitrary degree with states separating operation. In a similar way we can denote $P_S^{(i)} = P_D^{(i)} + P_E^{(i)}$ and find that the unitary evolution \hat{U} in Eq. (2) just transfers the states $|\Psi_i\rangle_A^{\otimes M} |P_0\rangle$ into the states $1/\sqrt{P_S^{(i)}}(\sqrt{\gamma_i}|\varphi_i\rangle_A + \sum_j t_{ij}|\varphi_j\rangle_A)|P_0\rangle$ with transfer probability $P_S^{(i)}$ and realize states separating operate. We can give $P_D = P_S P_H$, and $P_E = P_S(1 - P_H)$,

where P_H is the Helstrom limit for M initial states. With these representations, inequality (13) yields

$$P_E \geq \frac{1}{2} \left(P_S - \sqrt{P_S^2 - (P_S - P_{IDP})^2} \right) \quad \text{if } P_S \geq P_{IDP}. \quad (18)$$

This inequality is just that which has been obtained by Chefles and Barnett [9] when $M = 1$, which gives a bound on the error probability P_E given a fixed value of the probability P_S and it is equivalent to inequality (15).

So far we have constructed a general discrimination of the states secretly chosen from a certain set S given M initial copies of each state by a combination of unitary evolution together with measurements. We have derived the matrix inequality which describes the bound among three different discriminating results: correct, error and inconclusive. For different n -state, we give a condition which yields such n -state are linearly independent and find the result of Duan and Guo is just obtained in a special situation of our condition. For two states, we find the most general bound (inequality (9)) among the discriminating probabilities of correctly (P_D^+ , P_D^-), error (P_E^+ , P_E^-) and inconclusive (P_I^+ , P_I^-) and the former works are the results of different applications of our bound in a different situation.

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