INTEGRATING BY PARTS-Examples

The integration by parts procedure is just a reduction strategy (simplifying a difficult integral). It's based on the product rule for derivatives:

$$\tfrac{d}{dx}(uv) = u'v + uv' \Rightarrow \int \tfrac{d}{dx}(uv)dx = \int (u'v + v'u)dx$$
$$\Rightarrow uv = \int vdu + \int udv \Rightarrow \int udv = uv - \int vdu$$

In definite integral form: $\int_a^b udv = uv\big|_a^b - \int_a^b vdu$

General Strategy: Choose *u* (and hence *dv*) in such a manner that *vdu* is less complicated to integrate than *udv*

- Integrals involving $x^n e^{ax}$, $x^n \sin ax$, $x^n \cos ax$ are *recursive* (the parts strategy involves iteration). **Choose $u = x^n$ in these three cases, since *vdu* will involve a sinusoidal or exponential term, with a simpler x^{n-1} term.**

Example: Integrate $\int x^5 \sin axdx$

$$\int x^5 \sin axdx \Rightarrow u = x^5, dv = \sin axdx \therefore du = 5x^4 dx, v = -\tfrac{1}{a}\cos ax$$
$$\therefore \int x^5 \sin axdx = -\tfrac{1}{a}x^5 \cos ax + \tfrac{5}{a}\int x^4 \cos axdx$$

$$\int x^4 \cos axdx \Rightarrow u = x^4, dv = \cos axdx \therefore du = 4x^3 dx, v = \tfrac{1}{a}\sin ax$$
$$\therefore \int x^4 \cos axdx = \tfrac{1}{a}x^4 \sin ax - \tfrac{4}{a}\int x^3 \sin axdx$$

$$\int x^3 \sin axdx \Rightarrow u = x^3, dv = \sin axdx \therefore du = 3x^2 dx, v = -\tfrac{1}{a}\cos ax$$
$$\therefore \int x^3 \sin axdx = -\tfrac{1}{a}x^3 \cos ax + \tfrac{3}{a}\int x^2 \cos axdx$$

$$\int x^2 \cos axdx \Rightarrow u = x^2, dv = \cos axdx \therefore du = 2xdx, v = \tfrac{1}{a}\sin ax$$
$$\therefore \int x^2 \cos axdx = \tfrac{1}{a}x^2 \sin ax - \tfrac{2}{a}\int x \sin axdx$$

$$\int x\sin ax dx \Rightarrow u = x, dv = \sin ax dx \therefore du = dx, v = -\tfrac{1}{a}\cos ax$$

$$\therefore \int x\sin ax dx = -\tfrac{1}{a}x\cos ax + \tfrac{1}{a}\int \cos ax dx = -\tfrac{1}{a}x\cos ax + \tfrac{1}{a^2}\sin ax + C$$

Hence:

$$\int x^5 \sin ax dx = -\tfrac{1}{a}x^5\cos ax + \tfrac{5}{a}\left\{\tfrac{1}{a}x^4\sin ax - \tfrac{4}{a}\left\{-\tfrac{1}{a}x^3\cos ax + \tfrac{3}{a}\left[\tfrac{1}{a}x^2\sin ax - \tfrac{2}{a}\left(-\tfrac{1}{a}x\cos ax + \tfrac{1}{a^2}\sin ax\right)\right]\right\}\right\}$$

$$= -\tfrac{1}{a}x^5\cos ax + \tfrac{5}{a^2}x^4\sin ax + \tfrac{5\cdot4}{a^3}x^3\cos ax - \tfrac{5\cdot4\cdot3}{a^4}x^2\sin ax + \tfrac{5\cdot4\cdot3\cdot2}{a^5}x\cos ax - \tfrac{5\cdot4\cdot3\cdot2\cdot1}{a^6}\sin ax + C$$

$$= \tfrac{1}{a}\left(-x^5 + \tfrac{5\cdot4}{a^2}x^3 - \tfrac{5\cdot4\cdot3\cdot2}{a^4}x\right)\cos ax + \tfrac{1}{a^2}\left(5x^4 - \tfrac{5\cdot4\cdot3}{a^2}x^2 + \tfrac{5\cdot4\cdot3\cdot2\cdot1}{a^4}\right)\sin ax + C$$

Check:

$$\frac{d}{dx}\left[\tfrac{1}{a}\left(-x^5 + \tfrac{5\cdot4}{a^2}x^3 - \tfrac{5\cdot4\cdot3\cdot2}{a^4}x\right)\cos ax + \tfrac{1}{a^2}\left(5x^4 - \tfrac{5\cdot4\cdot3}{a^2}x^2 + \tfrac{5\cdot4\cdot3\cdot2\cdot1}{a^4}\right)\sin ax\right]$$

$$= -\tfrac{1}{a}a\sin ax\left(-x^5 + \tfrac{5\cdot4}{a^2}x^3 - \tfrac{5\cdot4\cdot3\cdot2}{a^4}x\right) + \tfrac{1}{a}\cos ax\left(-5x^4 + \tfrac{5\cdot4\cdot3}{a^2}x^2 - \tfrac{5\cdot4\cdot3\cdot2}{a^4}\right) + \tfrac{1}{a^2}a\cos ax\left(5x^4 - \tfrac{5\cdot4\cdot3}{a^2}x^2 + \tfrac{5\cdot4\cdot3\cdot2}{a^4}\right)$$

$$+ \tfrac{1}{a^2}\sin ax\left(5\cdot4x^3 - \tfrac{5\cdot4\cdot3\cdot2}{a^2}x\right)$$

$$= x^5\sin ax - \tfrac{5\cdot4}{a^2}x^3\sin ax + \tfrac{5\cdot4\cdot3\cdot2}{a^4}x\sin ax - \tfrac{5}{a}x^4\cos ax + \tfrac{5\cdot4\cdot3}{a^3}x^2\cos ax - \tfrac{5\cdot4\cdot3\cdot2}{a^5}\cos ax$$

$$+ \tfrac{5}{a}x^4\cos ax - \tfrac{5\cdot4\cdot3}{a^4}x^2\cos ax + \tfrac{5\cdot4\cdot3\cdot2}{a^5}\cos ax + \tfrac{5\cdot4}{a^2}x^3\sin ax - \tfrac{5\cdot4\cdot3\cdot2}{a^4}x\sin ax$$

$$= x^5\sin ax$$

Obviously this is a rather tedious procedure! Tabular Integration is more efficient:

u	***dv***
x^5 (+)	$\sin ax$
$5x^4$ (-)	$-\frac{1}{a}\cos ax$
$20x^3$ (+)	$-\frac{1}{a^2}\sin ax$
$60x^2$ (-)	$\frac{1}{a^3}\cos ax$
$120x$ (+)	$\frac{1}{a^4}\sin ax$
120 (-)	$-\frac{1}{a^5}\cos ax$
0	$-\frac{1}{a^6}\sin ax$

Hence reading from table:

$$\int x^5 \sin ax dx = -\tfrac{1}{a} x^5 \cos ax + \tfrac{5}{a^2} x^4 \sin ax + \tfrac{5\cdot 4}{a^3} x^3 \cos ax - \tfrac{5\cdot 4\cdot 3}{a^4} x^2 \sin ax$$
$$- \tfrac{5\cdot 4\cdot 3\cdot 2}{a^5} x \cos ax + \tfrac{5\cdot 4\cdot 3\cdot 2\cdot 1}{a^6} \sin ax + C$$
$$= \tfrac{1}{a}\left(-x^5 + \tfrac{5\cdot 4}{a^2} x^3 - \tfrac{5\cdot 4\cdot 3\cdot 2}{a^4} x\right)\cos ax + \tfrac{1}{a^2}\left(5x^4 - \tfrac{5\cdot 4\cdot 3}{a^2} x^2 + \tfrac{5\cdot 4\cdot 3\cdot 2\cdot 1}{a^4}\right)\sin ax + C$$

- Integrals involving $x^n \ln x$, $x^n \arctan x$, $x^n \arcsin x$ aren't *recursive* **Choose $dv = x^n$ in these three cases.**

Example: $\int x^3 \arctan x dx$

$$dv = x^3 dx \Rightarrow v = \tfrac{1}{4} x^4, u = \arctan x \Rightarrow du = \frac{dx}{1+x^2}$$

$$\therefore \int x^3 \arctan x dx = \tfrac{1}{4} x^4 \arctan x - \tfrac{1}{4} \int \frac{x^4}{1+x^2} dx$$

The integral on the right can simplified via polynomial division:

$$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1} \Rightarrow \int \frac{x^4}{1+x^2} dx = \int \left(x^2 - 1 + \frac{1}{x^2+1}\right) dx$$
$$= \tfrac{1}{3} x^3 - x + \arctan x + C$$
$$\therefore \int x^3 \arctan x dx = \tfrac{1}{4} x^4 \arctan x - \tfrac{1}{12} x^3 + \tfrac{1}{4} x - \tfrac{1}{4} \arctan x + C$$
$$= \tfrac{1}{4} \arctan x \left(x^4 - 1\right) - \tfrac{1}{4} x \left(\tfrac{1}{3} x^2 - 1\right) + C$$

Check:

$$\frac{d}{dx}\left(\tfrac{1}{4} x^4 \arctan x - \tfrac{1}{12} x^3 + \tfrac{1}{4} x - \tfrac{1}{4} \arctan x + C\right)$$
$$= x^3 \arctan x + \tfrac{1}{4} \cdot \frac{x^4}{1+x^2} - \tfrac{1}{4} x^2 + \tfrac{1}{4} - \tfrac{1}{4} \cdot \frac{1}{1+x^2}$$
$$= x^3 \arctan x + \tfrac{1}{4}\left\{\frac{x^4}{1+x^2} - x^2 + 1 - \frac{1}{1+x^2}\right\}$$
$$= x^3 \arctan x + \tfrac{1}{4}\left\{x^2 - 1 + \frac{1}{1+x^2} - x^2 + 1 - \frac{1}{1+x^2}\right\}$$
$$= x^3 \arctan x$$

- Integrals involving $e^{ax} \cos bx\ x$, $e^{ax} \sin bx$ aren't *recursive.* **One can choose u to be either the exponential or the sinusoidal term, it makes no difference in the amount of work.**

- Example (Exercise 55, §9.2)

Derive: $\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$

Choose (arbitrarily) $u = e^{ax}$. Then:

$$u = e^{ax} \Rightarrow du = ae^{ax}, dv = \sin bx dx \Rightarrow v = -\tfrac{1}{b}\cos bx$$
$$\therefore \int e^{ax}\sin bx = -\tfrac{1}{b}e^{ax}\cos bx + \tfrac{a}{b}\int e^{ax}\cos bx dx$$

$$u = e^{ax} \Rightarrow du = ae^{ax}, dv = \cos bx dx \Rightarrow v = \tfrac{1}{b}\sin bx$$
$$\therefore \int e^{ax}\cos bx dx = \tfrac{1}{b}e^{ax}\sin bx - \tfrac{a}{b}\int e^{ax}\sin bx dx$$

$$\therefore \int e^{ax}\sin bx = -\tfrac{1}{b}e^{ax}\cos bx + \tfrac{a}{b}\int e^{ax}\cos bx dx$$
$$= -\tfrac{1}{b}e^{ax}\cos bx + \tfrac{a}{b}\left(\tfrac{1}{b}e^{ax}\sin bx - \tfrac{a}{b}\int e^{ax}\sin bx dx\right)$$
$$= -\tfrac{1}{b}e^{ax}\cos bx + \tfrac{a}{b^2}e^{ax}\sin bx - \tfrac{a^2}{b^2}\int e^{ax}\sin bx dx$$
$$\therefore \left(1+\tfrac{a^2}{b^2}\right)\int e^{ax}\sin bx dx = \frac{e^{ax}(-b\cos bx + a\sin bx)}{b^2} + C$$
$$\int e^{ax}\sin bx dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{\left(1+\tfrac{a^2}{b^2}\right)b^2} = \frac{e^{ax}(a\sin bx - b\cos bx)}{b^2+a^2} + C$$

X.Trigonometric Substitutions:

- ***This method transforms a power of a binomial (usually difficult --and in some cases impossible-- to integrate by parts alone). The technique exploits the Pythagorean Identities:***

$\tan^2 x + 1 = \sec^2 x$ **$\cos^2 x + \sin^2 x = 1$**

...To reduce integrands of the form: **$u(x)^2 \pm a^2$** ***and:*** **$a^2 \pm u(x)^2$**

- ***There are obviously three cases. Inferrinng from Pythagorean Identities, substitute:***

$a^2 + u(x)^2 \Rightarrow u(x) = a\tan\theta$	**$a^2 - u(x)^2 \Rightarrow u(x) = a\sin\theta$**	**$u(x)^2 - a^2 \Rightarrow u(x) = a\sec\theta$**
		Formulae X.1a)-c)

***Note:* With no loss of generality, we could have just as well suggested *a*cotθ, *a*cosθ, *a*cscθ respectively for formulae X.1a)-c), but that wouldleave us with an extra, unnecessary ± sign complication.**

***Exercise* (# 8, 8.4)** $\int x^2 [x^2 - 4]^{-1/2} dx$

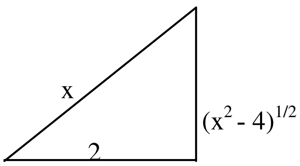
Solution: this warrants substitution X.1c) Let: **$x = 2\sec\theta$** ***then:*** **$dx = 2\sec\theta\tan\theta d\theta$**
SO:

$$\int x^2 [x^2 - 4]^{-1/2} dx = \int 4\sec^2\theta\, [4\sec^2\theta - 4]^{-1/2}\, (2\sec\theta\tan\theta d\theta) = \int 4\sec^2\theta\, [4\tan^2\theta]^{-1/2}\, (2\sec\theta\tan\theta d\theta)$$

$$= \int 4\sec^2\theta(\sec\theta d\theta) = 4\int \sec^3\theta d\theta = 4\,\{{}^1/_2 \sec\theta\tan\theta + {}^1/_2 \ln|\sec\theta + \tan\theta|\} + C$$

↑
Using Reduction Formula (or #36, Table)

$$= 2\sec\theta\tan\theta + 2\ln|\sec\theta + \tan\theta| + C = \frac{x(x^2 - 4)^{1/2}}{2} + \ln|x + (x^2 - 4)^{1/2}| + C$$



To rewrite in terms of x, note: **$x = 2\sec\theta \Rightarrow \cos\theta = {}^2/_x$**

Hence: $\sec\theta = {}^x/_2$ $\tan\theta = \frac{(x^2 - 4)^{1/2}}{2}$

<u>Exercise:</u> Use integration by Parts and Trig Subsitution to calculate in two ways (42, 8.3):

$$\int x\arctan x dx$$

1) Parts Alone (Not Recommended!)

Let $dv = xdx$ $u = \arctan x$ ***Then:*** $v = {}^1/_2\, x^2$ $du = (x^2 + 1)^{-1} dx$

$$\int x\arctan x dx = {}^1/_2\, x^2 \arctan x - {}^1/_2 \int x^2(x^2 + 1)^{-1} dx$$

↑
ad-hoc trick: $x^2 = x^2 + 1 - 1$ Hence: $x^2(x^2 + 1)^{-1} = [1 - (x^2 + 1)^{-1}]$

Hence: $\int x\arctan x dx = {}^1/_2\, x^2 \arctan x - {}^1/_2 \int [1 - (x^2 + 1)^{-1}]dx =$ **${}^1/_2\, x^2 \arctan x - {}^x/_2 + {}^1/_2 \arctan x + C$**

2) Trig Substitution (More Systematic, no ad-hoc tricks)

Let $x = \tan\theta$ $dx = \sec^2\theta d\theta$ and therefore $\theta = \arctan x$

Hence: $\int x\arctan x dx =$ Hence: $\int \theta\, (\tan\theta\sec^2\theta d\theta) = {}^1/_2\, \theta\tan^2\theta - \int {}^1/_2 \tan^2\theta d\theta$

↑ ↑
u dv (Must be evaluated by parts)

$$= {}^1/_2\, \theta\tan^2\theta - {}^1/_2 \int (\sec^2\theta - 1)d\theta = {}^1/_2\, \theta\tan^2\theta - {}^1/_2 \tan\theta + {}^1/_2\, \theta + C$$

$= {}^1/_2\, x^2 \arctan x - {}^x/_2 + {}^1/_2 \arctan x + C$

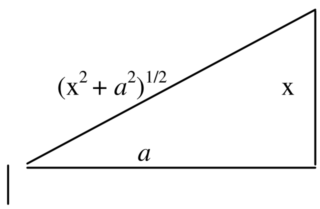
Exercise: ***Calculate***: $\int (x^2 + a^2)^{-n}\ dx$

Let $x = a\tan\theta$ then: $dx = a\sec^2\theta\ d\theta$ Hence: $\int (x^2 + a^2)^{-n}\ dx = \int (a^2 \sec^2\theta)^{-n}\ a\sec^2\theta\ d\theta$

$$= a^{1-2n} \int \sec^{-2(n-1)}\theta\ d\theta = a^{1-2n} \left\{ \frac{\sec^{(-2n+1)}\theta \sin\theta}{(-2n+1)} + \frac{(-2n\ )}{(-2n+1)} \int \sec^{(-2n)}\theta d\theta \right\}$$

↑

Using # 38, Table with "n" = -2n + 2



$x = a\tan\theta \Rightarrow \tan\theta = x/a$

$\Rightarrow \sec\theta = (x^2 + a^2)^{1/2}/a$

$\Rightarrow \sin\theta = x / (x^2 + a^2)^{1/2}$

$\Rightarrow \sec^{(-2n+1)}\theta \sin\theta = x a^{2n-1} (x^2 + a^2)^{-n}$

$\Rightarrow \sec^{(-2n)}\theta d\theta = a^{2n}\ (x^2 + a^2)^{-n-1}$

Hence: $\int (x^2 + a^2)^{-n}\ dx = \frac{x(x^2 + a^2)^{-n}}{(1 - 2n)} + \frac{2na}{(2n - 1)} \int (x^2 + a^2)^{-n-1}\ dx$

RATIONALIZING SUBSTITUTIONS

- ***For any rational expression*** $[g(x)]^{1/q}$**, set:** $[u(x)]^{q} = g(x)$
- ***For any <u>denominator</u> terms of the form Asinx ± Bcosx, let u(x) = tan(x/2)***

<u>Example (Type 1)</u> ***Integrate:*** $\int [x+1][(x - 2)^{1/2}x]^{-1}\ dx$ ***(#18, text)***

Let $u^2 = x - 2$, then: $x = u^2 + 2$, $dx = 2udu$

$$\int [x+1][(x - 2)^{1/2}x]^{-1}\ dx = \int [u^2 + 3][u(u^2 + 2)]^{-1}\ 2udu$$

<u>Example (Type 2) : (#28,text)</u>* *Integrate: $\int [\,1 + \sin x + \cos x\,]^{-1}\, dx$

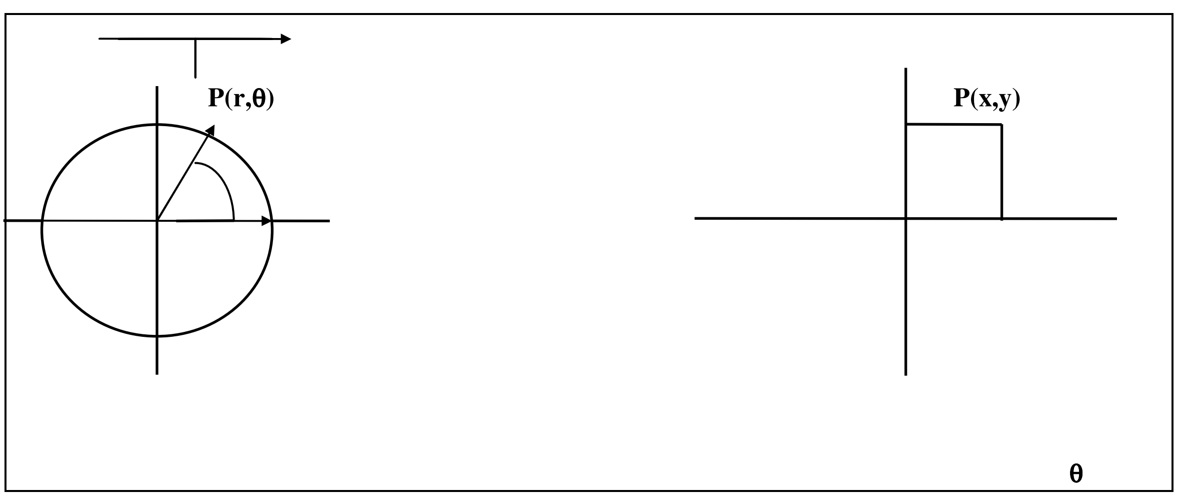
Let : $u = \tan(x/2)$ *Then:* $\sin(x/2) = u/[1 + u^2]^{1/2}$ $\qquad \cos(x/2) = 1/[1 + u^2]^{1/2}$

$\sin x = 2\sin(x/2)\cos(x/2) = 2u/[1 + u^2]$

$\cos x = \cos^2(x/2) - \sin^2(x/2) = (1 - u^2)/\,[1 + u^2]$

$du = (1/2)\, \sec^2 (x/2)dx \Rightarrow dx = 2\cos^2(x/2)du = [2/(1 + u^2)]du$

XIII Polar Coordinates



- **Non-uniqueness** $P(r,\theta) = P(-r, \theta + (2n+1)\pi) = P(r, \theta + 2n\pi)$

- **Transformation Rules:**

$x(r,\theta) = r\cos\theta$	$\theta = \tan^{-1}(y/x)$
$y(r,\theta) = r\sin\theta$	$r = (x^2 + y^2)^{1/2}$
Polar → Cartesian	***Cartesian → Polar***

- **In Cartesian Coordinates:** $y = f(x)$ $\quad$ $r = f(\theta)$

  ↑ ↑ $\quad$ ↑ ↑

  Dep. V Ind. Var $\quad$ ***Dep Var. Ind. Var.***

  <u>Functions</u>

<u>Example (48, 9.3)</u> Write the equation in polar coordinates: $(x^2 + y^2)^2 = x^2 - y^2$

<u>Example (58, 9.3)</u> Write the equation in Cartesin coordinates: $r = \sin\theta$, ***identify curve***