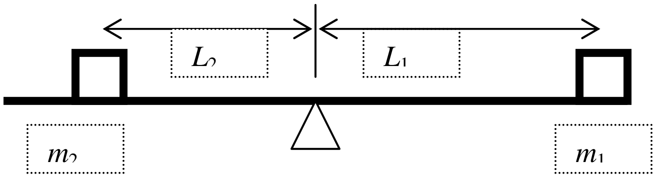
MA261 **Calculus I: Notes for October 18**
Fall 2007 Kallfelz

CENTROIDS & APPLICATIONS INVOLVING WORK AND PRESSURE

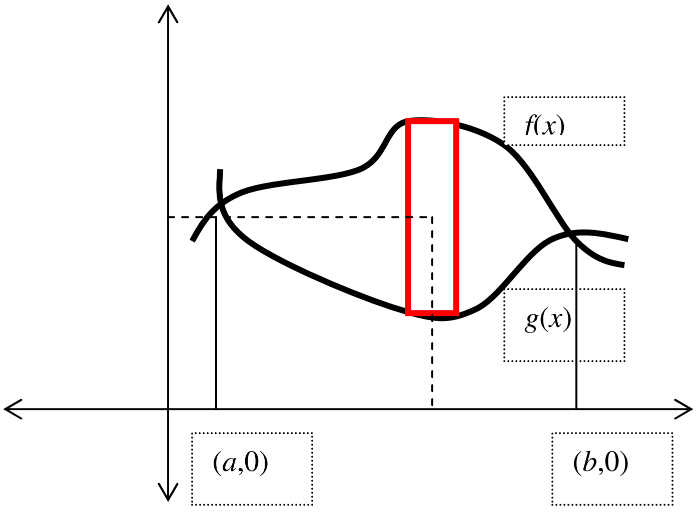
- Moments

The idea of moment goes back to Archimedes' Lever Principle. For masses m_1 & m_2 to balance on the lever (see illustration), the following ratios must equal each other: $\frac{m_1}{m_2} = \frac{L_2}{L_1}$



…from which the expressions: $m_1 L_1 = m_2 L_2$ can be derived. These quantities are known as *moments* (the product of a mass with the distance from its pivot.)

Now consider a 'laminar' (a region of constant density and thickness) surface whose face is bordered above by $f(x)$ and below by
$g(x)$:



The rectangle of width dx has a centroid whose y-coordinate is located at its center, i.e.: $\bar{y} = \frac{1}{2}(f(x) + g(x))$. So the moment of the rectangle about the x-axis is: $M_x^{(r)} = \bar{y}m^{(r)}$, where $m^{(r)}$ is the mass of the rectangle. Since the material has constant density ρ, the rectangle's mass is: $\rho L w$, where $L = (f(x) - g(x))$ is the rectangle's length and $w = dx$ is the rectangle's width. So:

$$M_x^{(r)} = \bar{y}m^{(r)} = \tfrac{1}{2}(f(x) + g(x))\rho(f(x) + g(x))dx = \frac{\rho}{2}(f(x) + g(x))(f(x) - g(x))dx$$

Hence the *total* moment of the object about the x-axis is:

$$M_x = \int_a^b M_x^{(r)} = \tfrac{\rho}{2}\int_a^b (f(x) + g(x))(f(x) - g(x))dx$$

Similary, in the case of $M_x^{(r)} = \bar{x}m^{(r)}$, then $\bar{x} = x$, simply the distance of the rectangle's center from the y-axis (which of course is simply the x-coordinate of the location of its center).

Hence the *total* moment of the object about the y-axis is:

$$M_y = \int_a^b M_y^{(r)} = \tfrac{\rho}{2}\int_a^b x(f(x) - g(x))dx$$

Then object's *mass* of course is simply: $M = \rho\int_a^b (f(x) - g(x))dx$. Hence the object's y-coordinate for its center of mass is :

$$\bar{y} = \frac{M_x}{M} = \frac{\tfrac{\rho}{2}\int_a^b (f(x) + g(x))(f(x) - g(x))dx}{\rho\int_a^b (f(x) - g(x))dx} = \frac{\tfrac{1}{2}\int_a^b (f(x) + g(x))(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}$$

Note1: The above formula presupposes a *constant* density. (Hence the density ρ is treated as a constant, and pulled out of the integrals above and below.) If this weren't true, then ρ can be expressed as a function of x, i.e.: $\rho = \rho(x)$, and must be kept inside the integrals. **Even then, this is not the most general case, since $\rho = \rho(x)$ depends *only* on x.** The most general case would be if $\rho = \rho(x,y)$ which is a multi-variate (Calculus III) problems

$$\bar{y}=\frac{M_x}{M}=\frac{\frac{1}{2}\int_a^b \rho(x)(f(x)+g(x))(f(x)-g(x))dx}{\int_a^b \rho(x)(f(x)-g(x))dx}$$

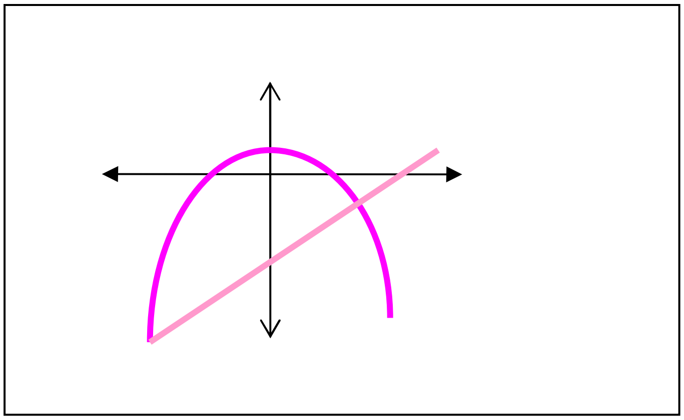
And the *x*- coordinate for the object's center of mass is:

$$\bar{x}=\frac{M_y}{M}=\frac{\rho\int_a^b x(f(x)-g(x))dx}{\rho\int_a^b (f(x)-g(x))dx}=\frac{\int_a^b x(f(x)-g(x))dx}{\int_a^b (f(x)-g(x))dx}$$

And in the more general case (non-uniform density):

$$\bar{x}=\frac{M_y}{M}=\frac{\int_a^b \rho(x)x(f(x)-g(x))dx}{\int_a^b \rho(x)(f(x)-g(x))dx}$$

- **Example:** Find the centroid for a laminar surface of constant density ρ bounded above by $f(x)=2-x^2$ and below by $g(x)=x-2$.



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$g(x)=f(x)\Rightarrow 2-x^2=x-2\Rightarrow x^2+x-4=0$

$\therefore x_{1,2}=\frac{-1\pm\sqrt{1+16}}{2}=\frac{1}{2}\left(-1\pm\sqrt{17}\right)$

First calculate the mass: $M = \int_{x_1}^{x_2} \rho\left(f(x) - g(x)\right)dx = \rho \int_{-\frac{1}{2}\left(1+\sqrt{17}\right)}^{\frac{1}{2}\left(-1+\sqrt{17}\right)} \left[\left(2 - x^2\right) - (x-2)\right]dx$

$$\therefore M = \int_{x_1}^{x_2} \rho dA = \rho \int_{-\frac{1}{2}\left(1+\sqrt{17}\right)}^{\frac{1}{2}\left(-1+\sqrt{17}\right)} \left[4 - x - x^2\right]dx = \rho\left(4x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right)\Big|_{-\frac{1}{2}\left(1+\sqrt{17}\right)}^{\frac{1}{2}\left(-1+\sqrt{17}\right)}$$

$$= \rho x\left(4 - \tfrac{1}{2}x - \tfrac{1}{3}x^2\right)\Big|_{-\frac{1}{2}\left(1+\sqrt{17}\right)}^{\frac{1}{2}\left(-1+\sqrt{17}\right)}$$

$$= \rho\left\{\left(\tfrac{1}{2}\left(-1+\sqrt{17}\right)\left[4 - \tfrac{1}{4}\left(-1+\sqrt{17}\right) - \tfrac{1}{12}\left(-1+\sqrt{17}\right)\right] + \tfrac{1}{2}\left(1+\sqrt{17}\right)\left[4 + \tfrac{1}{4}\left(1+\sqrt{17}\right) - \tfrac{1}{12}\left(1+\sqrt{17}\right)^2\right]\right)\right\}$$

$$= 46.7286\rho$$

The moment around the *x*-axis:

$$M_x = \tfrac{\rho}{2}\int_a^b \left(f(x) + g(x)\right)\left(f(x) - g(x)\right)dx = \tfrac{\rho}{2}\int_{x_1}^{x_2} \left[\left(2 - x^2\right) + (x-2)\right]\left[4 - x - x^2\right]dx$$

$$= \tfrac{\rho}{2}\int_{x_1}^{x_2} \left(x - x^2\right)\left(4 - x - x^2\right)dx = \tfrac{\rho}{2}\int_{x_1}^{x_2} \left[\left(4x - x^2 - x^3\right) - \left(4x^2 - x^3 - x^4\right)\right]dx$$

$$= \tfrac{\rho}{2}\int_{x_1}^{x_2} \left(4x - 5x^2 + x^4\right)dx = \tfrac{\rho}{2}\left(2x^2 - \tfrac{5}{3}x^3 + \tfrac{1}{5}x^5\right)\Big|_{x_1}^{x_2} = \rho x^2\left(1 - \tfrac{5}{6}x + \tfrac{1}{10}x^3\right)\Big|_{-\frac{1}{2}\left(1+\sqrt{17}\right)}^{\frac{1}{2}\left(-1+\sqrt{17}\right)}$$

$$= -33.2597\rho$$

(This answer tells us that more of the mass is concentrated below the *x*-axis, rather than above it, which makes sense from the shape of the figure)

The moment around the *y*-axis:

$$M_x = \rho\int_a^b x\left(f(x) - g(x)\right)dx = \rho\int_{x_1}^{x_2} \left[4x - x^2 - x^3\right]dx$$

$$\rho\int_{x_1}^{x_2} \left(4x - x^2 - x^3\right)dx = \rho\left(2x^2 - \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4\right)\Big|_{x_1}^{x_2} = \rho x^2\left(2 - \tfrac{1}{3}x + \tfrac{1}{4}x^2\right)\Big|_{-\frac{1}{2}\left(1+\sqrt{17}\right)}^{\frac{1}{2}\left(-1+\sqrt{17}\right)}$$

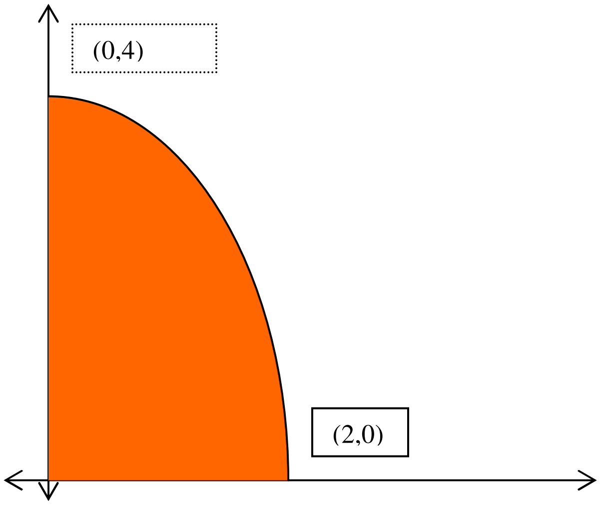
$$= -24.395\rho$$

So: $\bar{y} = \dfrac{M_x}{M} = \dfrac{-33.2597\rho}{46.7286\rho} = -0.7118 \quad \bar{x} = \dfrac{M_y}{M} = \dfrac{-24.395\rho}{46.7286\rho} = -0.52206$

Which locates the object's center of mass in the 4th quadrant. Note how these answers correspond with what's expected, since much of the object's mass is located there. It has a bottom left tip with coordinates: $\left(-\frac{1}{2}\left(1+\sqrt{17}\right), f\left(-\frac{1}{2}\left(1+\sqrt{17}\right)\right)\right)=$ (-2.56155, -4.56155), and a top right tip with coordinates: $\left(\frac{1}{2}\left(-1+\sqrt{17}\right), f\left(\frac{1}{2}\left(-1+\sqrt{17}\right)\right)\right)=$ (1.561553, -0.43845).

- Example (Non-uniform density)

  Suppose we have a laminar region sitting above the x-axis described by the inverted parabola: $f(x)=4-x^2$, and bounded by x = 0, y = 0, with density function: $\rho(x)=k(2-x)$ (i.e. mass minimal at the rightmost tip)



Calculating mass:

$$M=\int_0^2 \rho(x)(f(x)-0)dx=k\int_0^2 (2-x)\left(4-x^2\right)dx=k\int_0^2\left(8-4x-2x^2+x^3\right)dx$$

$$=k\left(8x-2x^2-\tfrac{2}{3}x^3+\tfrac{x^4}{4}\right)\Big|_0^2=kx\left(8-2x-\tfrac{2}{3}x^2+\tfrac{1}{4}x^3\right)\Big|_0^2=2k\left(8-4-\tfrac{8}{3}+2\right)=\tfrac{20}{3}k$$

Calculating moment (about x-axis)

$$M_x=\tfrac{k}{2}\int_0^2 (f(x)+0)(f(x)-0)\rho(x)dx=\tfrac{k}{2}\int_0^2\left(4-x^2\right)^2(2-x)dx$$

One might be tempted to perform a *u*-substitution, since the term in the parenthesis $4-x^2$ has derivative -2*x*. However note that the second term: (2 – *x*) isn't so easily related to *du*, since the extra constant term 2 is *added* to –*x* (not multiplied). There are, however, several options to simplify the integral:

a.) Split the integral into two by distributing using the (2 – *x*) terms
b.) Define: *w* = 2-*x* and re-write everything in terms of *w* and *then* do a *u*-substitution.

It turns out that option a) is the most suitable:

$$M_x = \tfrac{k}{2}\int_0^2 (f(x)+0)(f(x)-0)\rho(x)dx = \tfrac{k}{2}\int_0^2 (4-x^2)^2(2-x)dx$$

$$= \tfrac{k}{2}\cdot 2\int_0^2 (4-x^2)^2\,dx - \tfrac{k}{2}\int_0^2 (4-x^2)^2\,xdx$$

$$= k\int_0^2 (16-8x^2+x^4)dx + \tfrac{k}{4}\int_{u(0)}^{u(2)} u^2 du$$

The second integral was transformed via: $u = 2-x^2 \Rightarrow du = -2xdx \Rightarrow xdx = -\tfrac{1}{2}du$

$$k\left(16x - \tfrac{8}{3}x^3 + \tfrac{1}{5}x^5\right)\Big|_0^2 + \tfrac{k}{4}\cdot\tfrac{1}{3}u^3\Big|_4^0 = kx\left(16 - \tfrac{8}{3}x^2 + \tfrac{1}{5}x^4\right)\Big|_0^2 + \tfrac{k}{12}u^3\Big|_4^0$$

$$== 2k\left(16 - \tfrac{32}{3} + \tfrac{16}{5}\right) - \tfrac{64}{12}k = 32k\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) - \tfrac{16}{3}k = 32k\left(\tfrac{8}{15}\right) - \tfrac{16}{3}k$$

$$= \tfrac{256-90}{15}k = \tfrac{166}{15}k$$

Calculating moment about *y*-axis:

$$M_y = \int_0^2 x(f(x)-0)\rho(x)dx = k\int_0^2 x(4-x^2)(2-x)dx = k\int_0^2 (8x-4x^2-2x^3+x^4)dx$$

$$= k\left(4x^2 - \tfrac{4}{3}x^3 - \tfrac{1}{2}x^4 + \tfrac{1}{5}x^5\right)\Big|_0^2 = kx^2\left(4 - \tfrac{4}{3}x - \tfrac{1}{2}x^2 + \tfrac{1}{5}x^3\right)\Big|_0^2 = 4k\left(4 - \tfrac{4}{3} - 2 + \tfrac{8}{5}\right)$$

$$= 4k\left(\tfrac{34}{15}\right) = \tfrac{136k}{15}$$

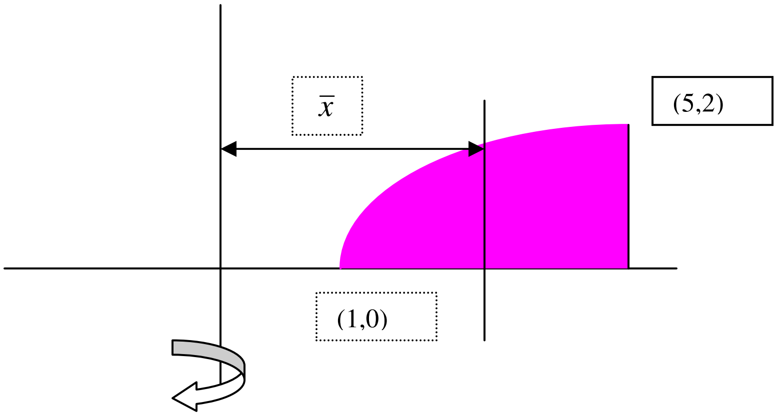
So: $\bar{x} = \dfrac{M_y}{M} = \dfrac{\frac{136}{15}k}{\frac{20}{3}k} = \tfrac{136}{15}\cdot\tfrac{3}{20} = \tfrac{68}{50} = \tfrac{34}{25}$ $\bar{y} = \dfrac{M_x}{M} = \dfrac{\frac{166}{15}k}{\frac{20}{3}k} = \tfrac{166}{15}\cdot\tfrac{3}{20} = \tfrac{83}{50}$

The **Theorem of Pappus** is one interesting way of calculating volumes of solids, when generated by rotating an area around a line that doesn't intersect the area anywhere (i.e. when generating a torus). The Theorem states that the volume should be:

$V = 2\pi r A$ where: A is the area of the figure and r is the distance away from its centroid to the axis of rotation.

- Example (42, p. 355)

The region $y = \sqrt{x-1}$ bounded by $x = 5$ and $y = 0$ and rotated around the y – axis:



As the drawing indicates, we need to obtain the x-coordinate of the centroid: $\overline{x} = \dfrac{M_y}{M}$

Calculating M_y: $M_y = \rho\int_1^5 (f(x)-0)x\,dx = \rho\int_0^2 \sqrt{x-1}\,x\,dx$

This is evaluated via a u-substitution: $u = x-1 \Rightarrow du = dx, x = u+1$

$$M_y = \rho\int_1^5 (f(x)-0)x\,dx = \rho\int_1^5 \sqrt{x-1}\,x\,dx = \rho\int_{u(1)}^{u(5)} u^{1/2}(u+1)\,du = \rho\int_0^4 \left(u^{3/2}+u^{1/2}\right)du$$

$$= \rho\left(\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right)\Big|_0^4 = 2u^{3/2}\rho\left(\tfrac{1}{5}u + \tfrac{1}{3}\right)\Big|_0^4 = 16\rho\left(\tfrac{1}{5}+\tfrac{1}{3}\right) = \tfrac{128}{15}\rho$$

Calculating M: $M = \rho\int_1^5 (f(x)-0)\,dx = \rho\int_0^2 \sqrt{x-1}\,dx$

$$M = \rho\int_{1}^{5}\left(f(x)-0\right)x\,dx = \rho\int_{1}^{5}\sqrt{x-1}\,dx = \rho\int_{u(1)}^{u(5)} u^{1/2}\,du$$

$$= \tfrac{2}{3}\rho u^{3/2}\Big|_{0}^{4} = \tfrac{2}{3}\cdot 8\rho = \tfrac{16}{3}\rho$$

So: $$\bar{x} = \frac{M_y}{M} = \frac{\frac{128}{15}\rho}{\frac{16}{3}\rho} = \tfrac{128}{15}\cdot\tfrac{3}{16} = \tfrac{8}{5}$$

The area A, of course, is just $A = \dfrac{M}{\rho} = \tfrac{16}{3}$

So, according to the Theorem of Pappus: $V = 2\pi\bar{x}A = 2\pi\cdot\tfrac{8}{5}\cdot\tfrac{16}{3} = \tfrac{256}{15}\pi$

EXPOENTIAL AND LOGARITHMIC FUNCTIONS

Recall:

For any exponential function $f(x) = a^x$, its inverse function $f^{-1}(x) = \log_a x$

By property of inverses: $f\left(f^{-1}(x)\right) = x = f^{-1}\left(f(x)\right)$

So in the case of exponential and logs:

$$a^{\log_a x} = x = \log_a\left(a^x\right)$$

(by virtue of the composition property of a function with its inverse as defined above)

Recall: (For any base *a* > 0)

- **a.)** $a^0 = 1$
- **b.)** $a^x \cdot a^y = a^{(x+y)}$
- **c.)** $a^x / a^y = a^x \cdot a^{-y} = a^{(x-y)}$
- **d.)** $\left(a^x\right)^y = a^{xy}$

From the above, we can derive:

- **a.)** $\log_a(1) = \log_a\left(a^0\right) = 0$
- **b.)** $\log_a\left(a^x \cdot a^y\right) = \log_a\left(a^{x+y}\right) = x + y = \log_a\left(a^x\right) + \log_a\left(a^y\right)$

  Hence renaming: $u = a^x, v = a^y$ we get the result:

$$\log_a(uv) = \log_a u + \log_a v$$

c.) $\log_a\left(\dfrac{a^x}{a^y}\right) = \log_a\left(a^{x-y}\right) = x - y = \log_a\left(a^x\right) - \log_a\left(a^y\right)$

Hence renaming: $u = a^x, v = a^y$ we get the result:

$$\log_a\left(\frac{u}{v}\right) = \log_a u - \log_a v$$

d.) $\log_a\left(\left(a^x\right)^y\right) = \log_a\left(a^{xy}\right) = xy = yx = y\log_a a^x$

Hence renaming: $u = a^x$ we get the result:

$$\log_a\left(u^y\right) = y\log_a u$$

Hence the commonly known algebraic properties of logarithms (as summarized above) all derive from the algebra of exponentiation + acting on exponential with the inverse function (definition of logarithm).

Consider the **natural base**[1] $e = \lim_{n\to\infty}\left(1 + \dfrac{1}{x}\right)^x = \lim_{u\to\infty}\left(1+u\right)^{\frac{1}{u}}$

The exponential and log functions are defined:

$$f(x) = e^x \qquad\qquad f^{-1}(x) = \log_e x \equiv \ln x$$

Where, by composition of inverses: $e^{\ln x} = x = \ln\ \left(e^x\right)$

In calculus, the natural logarithm is defined via a **definite integral:** $\ln x = \displaystyle\int_1^x \frac{dt}{t}$

Recall from the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)dt = \frac{d}{dx}\left(F(x) - F(a)\right) = \frac{d}{dx}F(x) - \frac{d}{dx}F(a) = f(x)$$

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[1] As Tristan correctly pointed out in class (10/18/07), this number was discovered by bankers, when considering the limit of the compounding interest formula in terms of # of times the interest can compound (the continuously compounding limit). For finite # of compounding periods (where *k* = 1 for yearly compounding interest, k=12 for monthly compounding interest, k=365 for daily compounding interest, we have: $A(t) = A_0\left(1 + \frac{r}{k}\right)^{kt}$, where *r* is the (yearly) interest rate, *A*(*t*) is the amount after *t* years, and A_0 is the principal amount. Then:

$$\lim_{k\to\infty} A(t) = \lim_{k\to\infty} A_0\left(1 + \tfrac{r}{k}\right)^{kt} = A_0\left[\lim_{k\to\infty}\left(1 + \tfrac{r}{k}\right)^k\right]^t = A_0 e^{rt}$$

Therefore: $$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x} \qquad \textbf{(A)}$$

Or, by the Chain Rule: $$\frac{d}{dx}\ln(u(x)) = \frac{d}{du}\ln u\,\frac{du}{dx} = \frac{1}{u(x)}u'(x) \qquad \textbf{(B)}$$

Formulae (A) and (B) have their associated antiderivative counterparts:

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad \int_a^b \frac{1}{x}\,dx = \ln|x|\Big|_a^b = \ln|b| - \ln|a| = \ln\left|\frac{b}{a}\right| \qquad \textbf{(C)}$$

$$\int \frac{1}{u}\,du = \ln|u| + C \qquad \int_{u(a)}^{u(b)} \frac{1}{u}\,du = \ln|u|\Big|_{u(a)}^{u(b)} = \ln|u(b)| - \ln|u(a)| = \ln\left|\frac{u(b)}{u(a)}\right| \qquad \textbf{(D)}$$

Note: The absolute values are necessary because the domain of the logarithmic function are the positive real numbers. In other words ln(x) DNE for all $x \le 0$. To see why, note that for *any* x (negative or positive), $e^x > 0$. Hence the logarithm tells us by what power we must raise e to in order to get, let's say, y. I.e., if $y = e^x$, then ln(y) = x. But because, for *any* x, $e^x > 0$, then $y > 0$. Therefore the argument of ln is always positive.

Given formulae (A)-(D) we can derive derivative and integral formulae for the (natural) exponential function. Recall: $e^{\ln x} = x = \ln\left(e^x\right)$. We seek an expression for $\frac{d}{dx}e^x$:

Note: $$x = \ln\left(e^x\right) \Rightarrow \frac{d}{dx}x = 1 = \frac{d}{dx}\ln\left(e^x\right) = \frac{1}{e^x}\frac{d}{dx}e^x$$

The right-hand side was evaluated via (Formula **(B)**)

Hence isolating $\frac{d}{dx}e^x$ on the right hand side of the expression we obtain:

$$\frac{d}{dx}e^x = e^x \qquad \textbf{(E)}$$

Formula **(E)** tells us why exponential functions grow so fast. They're equal to their own rates of change! (Recall on the other hand that in the case of a power-form x^n, its rate of change is always one degree less)

The chain rule generalization of **(E)** is: $\frac{d}{dx}e^{u(x)} = e^{u(x)}\frac{d}{dx}u = e^u u'$ **(F)**

Hence the associated antiderivative expressions are:

$$\int e^x dx = e^x + C \qquad \int_a^b e^x dx = e^x \Big|_a^b = e^b - e^a \qquad \textbf{(G)}$$

$$\int e^u du = e^u + C \qquad \int_{u(a)}^{u(b)} e^u du = e^u \Big|_{u(a)}^{u(b)} = e^{u(b)} - e^{u(a)} \qquad \textbf{(H)}$$

Now, based the above procedure, we can derive derivative and antiderivative results for *any* base a, i.e. for functions $f(x) = a^x$, and its inverse function $f^{-1}(x) = \log_a x$

To derive an expression for $\frac{d}{dx}a^x$, observe: $a^x = [\ln(e^a)]^x = (e^{\ln a})^x$ (by composition property: $e^{\ln x} = x = \ln\ (e^x)$, hence: $e^{\ln a} = a = \ln\ (e^a)$)

So we can re-write: $\frac{d}{dx}a^x = \frac{d}{dx}(e^{\ln a})^x = \frac{d}{dx}(e^{\ln ax})$

Using Formula **(F):** $\frac{d}{dx}(e^{\ln ax}) = e^{\ln ax}\frac{d}{dx}(\ln ax) = e^{\ln ax}\ln a$

.where $u(x) = \ln ax$. But note that $\ln a$ is just a constant, hence: $\frac{d}{dx}(\ln ax) = \ln a \frac{d}{dx}x = \ln a$

But as discussed above, $e^{\ln ax} = (e^{\ln a})^x = a^x$

Hence: $\frac{d}{dx}a^x = a^x \ln a = (\ln a)a^x$ **(I)**

Note that in the special case where $a = e$: $\frac{d}{dx}e^x = e^x \ln e = (\ln e)e^x = e^x$
(recovering **(E)**)

The chain rule version of **(I)** is: $\frac{d}{dx}a^{u(x)} = (\ln a)a^{u(x)}u'(x)$ **(J)**

The antiderivative expressions are:

$$\int a^x dx = \frac{1}{\ln a} a^x + C \qquad \int_c^d a^x dx = \frac{1}{\ln a} a^x \Big|_c^d = \frac{1}{\ln a}\left(a^d - a^c\right) \quad \textbf{(K)}$$

$$\int a^u du = \frac{1}{\ln a} a^u + C \qquad \int_{u(c)}^{u(d)} a^u du = \frac{1}{\ln a} a^u \Big|_{u(c)}^{u(d)} = \frac{1}{\ln a}\left(a^{u(d)} - a^{u(c)}\right) \quad \textbf{(L)}$$

Formulae **(A) – (L)** span the framework for the calculus of exponential and logarithmic functions.

- Example (#18, 369)

$$\frac{d}{dx}\left(\frac{e^{x/2}}{\sqrt{x}}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}} e^{\frac{x}{2}}\right) = \left(\frac{d}{dx} x^{-1/2}\right) e^{x/2} + x^{-1/2}\left(\frac{d}{dx} e^{x/2}\right)$$

(using the product rule)

Then: $\left(\frac{d}{dx} x^{-1/2}\right) e^{x/2} + x^{-1/2}\left(\frac{d}{dx} e^{x/2}\right) = \left(-\frac{1}{2} x^{-\frac{3}{2}}\right) e^{x/2} + x^{-1/2}\left(e^{x/2} \cdot \frac{1}{2}\right)$

(using formula **(F)** to differentiate the exponential)

Simplifying:

$$\left(-\frac{1}{2} x^{-\frac{3}{2}}\right) e^{x/2} + x^{-1/2}\left(e^{x/2} \cdot \frac{1}{2}\right) = \frac{1}{2} e^{x/2}\left(-x^{-3/2} + x^{1/2}\right)$$
$$= \frac{1}{2} e^{x/2} x^{-3/2}\left(-1 + x^2\right) = \frac{e^{x/2}\left(x^2 - 1\right)}{x\sqrt{x}}$$

- Example (#29, 369)

$$g(x) = (1 + 2x)e^{4x} \Rightarrow g'(x) = \left(\frac{d}{dx}(1 + 2x)\right) e^{4x} + (1 + 2x)\left(\frac{d}{dx} e^{4x}\right)$$
$$= 2e^{4x} + (1 + 2x)4e^{4x} = e^{4x}(2 + 4 + 8x) = e^{4x}(6 + 8x)$$
$$\Rightarrow g''(x) = \left(\frac{d}{dx} e^{4x}\right)(6 + 8x) + e^{4x}\left(\frac{d}{dx}(6 + 8x)\right) = 4e^{4x}(6 + 8x) + 8e^{4x}$$
$$= e^{4x}(24 + 32x + 8) = e^{4x}(32 + 32x) = 32e^{4x}(1 + x)$$

- Example (45, 370)

$$\int_0^2 \left(x^2-1\right)e^{\left(x^3-3x+1\right)}dx \qquad \text{Let} \quad u(x)=x^3-3x+1 \Rightarrow \frac{du}{dx}=3x^2-3 \Rightarrow du=3\left(x^2-1\right)dx$$

Hence: $\left(x^2-1\right)dx=\frac{1}{3}du$

$$\therefore \int_0^2 \left(x^2-1\right)e^{\left(x^3-3x+1\right)}dx=\frac{1}{3}\int_{u(0)}^{u(2)} e^u\,du=\frac{1}{3}e^u\Big|_1^3=\frac{1}{3}\left(e^3-e^1\right)=\frac{e}{3}\left(e^2-1\right)$$