Lecture 5. Complete set of commuting observables, Complementary Observables, Uncertainty Relation, Friday, Sept. 9-Monday, Sept. 12

Suppose operator $[A, B] = 0$ and there is no degeneracy in the spectrum of A. Let us write

$$A|a_n\rangle = a_n|a_n\rangle \tag{34}$$

Then it is easy to show that $|a_n\rangle$ is also an eigenstate of B. Since $AB|a_n\rangle = BA|a_n\rangle$, or $AB|a_n\rangle = a_nB|a_n\rangle$, $B|a_n\rangle$ is an eigenstate of A with eigenvalue a_n. However, since there is no degeneracy, we have $B|a_n\rangle = b_n|a_n\rangle$, or $|a_n\rangle$ is also an eigenstate of B. We can also label the state by $|a_n, b_n\rangle$ using the simultaneous eigenvalues of A and B.

If, however, a_n is a degenerate eigenvalue of A, the eigenstates form a subspace of certain dimension d_n (which is called degeneracy). One can choose any orthogonal basis in this subspace to be independent eigenstates of eigenvalue a_n. On the other hand, $B|a_n\rangle$ belongs to the degenerate subspace. In other words, B is block diagonal in the basis formed by A eigenstates. If we diagonalize B and if B is not degenerate in the subspace, the eigenstates can be completely labelled by $|a_n, b_n\rangle$, which are simultaneous eigenstates of A and B.

If there is still degeneracy, one can find an additional operator C which commutes with both A and B. Then we can look for common eigenstates of A, B, and C: $|a_n, b_n, c_n\rangle$ which might have no degeneracy, and so on. If a set of operators commute with each other and are complete in the sense that all common states are non-degenerate, we call such a set the complete set of commuting observables (CSCO).

If two observables do not commute $[A, B] \neq 0$, we call them incompatible. *Two incompatible observables cannot have a common set of eigenstates. Or They cannot be diagonalized simultaneously.* One example is S_x and S_z for a spin-1/2 particle. Occasionally, however, they may have *some* common eigenstates.

One of the important features of quantum mechanics is interference which has its origin in the linear superposition principle. Consider a beam of particles going through a double-slit, projecting an interference pattern on a screen far behind. Clearly, particles develop transverse momentum after going through the slits. The interference pattern is a measurement of position z along the vertical direction at the screen, which is in some sense a measure of vertical momentum just behind the slits. This measurement is incompatible with the notion that through which slit a particle goes (the position of the particle). If we insist on knowing by performing some measurement, it is well-known that we will destroy the interference because position and momentum are incompatible.

The above notion of incompatible measurements can be generalized as follows. Suppose we start with a state $|\psi\rangle$, and we measure B, and after that, we measure C. Let us compute the probability of obtaining c disregarding whatever values of b we get, then we have,

$$\begin{aligned} P_{c|b} &= \sum_b |\langle c|b\rangle|^2 |\langle b|\psi\rangle|^2 \\ &= \sum_b \langle c|b\rangle\langle b|\psi\rangle\langle\psi|b\rangle\langle b|c\rangle \end{aligned} \quad (35)$$

However, if we do not measure B, the probability of getting c is,

$$\begin{aligned} P_c &= |\langle c|\psi\rangle|^2 \\ &= \sum_{b,b'} \langle c|b'\rangle\langle b'|\psi\rangle\langle\psi|b\rangle\langle b|c\rangle \end{aligned} \quad (36)$$

which differs from $P_{c|b}$ by the interference term. Thus if B and C are incompatible, a measurement of B will affect the subsequent measurement of C, and vice versa. The interference term vanishes only if $[B, C] = 0$!

The point can be seen in the example of spin measurement. If we start with a state $|+\rangle_x$ and measure S_x, we always get the same answer $\hbar/2$. On the other hand, if one measures S_z and then measure S_x, then we always have 50% chance of getting $s_x = -\hbar/2$. Therefore a measurement of an incompatible variable destroys the interference contribution.

A related phenomena is the uncertainty relation: Given any quantum mechanical state, one can measure either B and C independently. But the uncertainties in B and C are related in these independent measurements. Define $\Delta A = A - \langle A\rangle$, it is easy to show $\langle(\Delta A)^2\rangle = \langle A^2\rangle - \langle A\rangle^2$. The uncertainty relation says that

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A, B]\rangle|^2 \quad (37)$$

This can be proved using the Schwartz inequality $\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2$ by choosing $|\alpha\rangle = \Delta A|\psi\rangle$ and $|\beta\rangle = \Delta B|\psi\rangle$