Lecture 25, Clebsch-Gordan Coefficients, Friday, Nov. 4 and Monday, Nov. 7

Let us find all the states \(|jm⟩\) and construct the C-G coefficients. We assume \(j_1 \geq j_2\). We can picture all states in terms of a two-dimensional lattice with \(m_1\) in the \(x\) direction and \(m_2\) in \(y\) direction.

All states \(|m_1m_2⟩\) are eigenstates of \(J_z = J_{1z} + J_{2z}\) with eigenvalues \(m_1 + m_2\),

\[ J_z|m_1m_2⟩ = (m_1 + m_2)|m_1m_2⟩. \] (71)

The state with the highest \(m_j\) is \(j_1 + j_2\), and there is only one such state. This state must have \(j = j_1 + j_2\). Clearly if \(j\) is larger than this, then we must have states with \(m_j\) larger as well, and we don’t. Therefore, we write,

\[ |j = j_1 + j_2, m = j_1 + j_2⟩ = |m_1 = j_1, m_2 = j_2⟩ \] (72)

Therefore, we have the following C-G coefficient,

\[ ⟨j = j_1 + j_2, m = j_1 + j_2|j_1, m_2 = j_2⟩ = 1 \] (73)

Consider next the states with \(m = j_1 + j_2 - 1\). There are two such states \(|m_1 = j_1, m_2 = j_2 - 1⟩\) and \(|m_1 = j_1 - 1, m_2 = j_2⟩\). One combination of them must give \(|j = j_1 + j_2, m = j_1 + j_2 - 1⟩\) which can be obtained by applying \(J_−\) to \(|j = j_1 + j_2, m = j_1 + j_2⟩\),

\[ |j_1 + j_2, j_1 + j_2 - 1⟩ \sim J_−|j_1, j_2⟩ \]

\[ = \sqrt{\frac{j_2}{j_1 + j_2}}|j_1, j_2 - 1⟩ + \sqrt{\frac{j_1}{j_1 + j_2}}|j_1 - 1, j_2⟩ \] (74)

From these, we have the following C-G coefficients,

\[ ⟨j_1 + j_2, j_1 + j_2 - 1|j_1, j_2 - 1⟩ = \sqrt{\frac{j_2}{j_1 + j_2}}; \]

\[ ⟨j_1 + j_2, j_1 + j_2 - 1|j_1 - 1, j_2⟩ = \sqrt{\frac{j_1}{j_1 + j_2}} \] (75)

There is an orthogonal combination of \(|j_1, j_2 - 1⟩\) and \(|j_1 - 1, j_2⟩\),

\[ \sqrt{\frac{j_1}{j_1 + j_2}}|j_1, j_2 - 1⟩ - \sqrt{\frac{j_2}{j_1 + j_2}}|j_1 - 1, j_2⟩ \] (76)
Clearly there is some arbitrariness in choosing the phase here. We assume that the coefficient with maximum \(m_1\) is positive. The above state must have \(j = j_1 + j_2 - 1\). Therefore, we have another set C-G coefficients,

\[
\langle j_1 + j_2 - 1, j_1 + j_2 - 1 | j_1, j_2 - 1 \rangle = \sqrt{\frac{j_1}{j_1 + j_2}}; \\
\langle j_1 + j_2 - 1, j_1 + j_2 - 1 | j_1 - 1, j_2 \rangle = -\sqrt{\frac{j_2}{j_1 + j_2}}
\] (77)

We consider now states with \(m = j_1 + j_2 - 2\). In general, there are three such states, \(|j_1, j_2 - 2\rangle, |j_1 - 1, j_2 - 1\rangle, \) and \(|j_1 - 2, j_2\rangle\). We obtain two combinations of these states as \(|j = j_1 + j_2, m = j_1 + j_2 - 2\rangle, |j = j_1 + j_2 - 1, m = j_1 + j_2 - 2\rangle\) by applying \(J_-\) on states \(|j_1 + j_2, j_1 + j_2 - 1\rangle\) and \(|j_1 + j_2 - 1, j_1 + j_2 - 1\rangle\). The final orthogonal combination must correspond to the state \(|j_1 + j_2 - 2, j_1 + j_2 - 2\rangle\). Therefore, every time we go to a lower \(m\), we find a set of states with \(j = j_1 + j_2, \ldots, m + 1\) by applying the ladder operator \(J_-\) to the states with \(j = m + 1\). The final orthogonal combination defines the state with \(j = m\). This process goes on until \(m = j_1 - j_2\), beyond that, the number of \(m\) states no longer increases, and all eigenstates of \(J^2\) and \(J_z\) can be obtained this way from applying the ladder operator. We then find all states with \(j = j_1 + j_2, j_1 + j_2 - 1, \ldots, |j_1 - j_2|\), and the corresponding C-G coefficients as well.

With the above phase convention, the CG coefficients depend on the order of the angular momentum coupling. They have the following symmetry property,

\[
\langle j_1m_1j_2m_2jm \rangle = (-1)^{j_1-j_2}\langle j_2m_2j_1m_1jm \rangle.
\] (78)

Let us derive two recursion relations for the C-G coefficients. Applying \(J_\pm\) on \(|jm\rangle\), we have,

\[
J_\pm|jm\rangle = (J_{1\pm} + J_{2\pm}) \sum_{m_1, m_2} |m_1m_2\rangle \langle m_1m_2jm\rangle
\] (79)

Using the "famous" formula, we have,

\[
\sqrt{(j \mp m)(j \pm m + 1)|jm \pm 1\rangle} = \sum_{m'_1, m'_2} (\sqrt{(j_1 \mp m'_1)(j_1 \pm m'_1 + 1)|m'_1 \pm 1, m'_2\rangle} \\
+ \sqrt{(j_2 \mp m'_2)(j_2 \pm m'_2 + 1)|m'_1, m'_2 \pm 1\rangle})\langle m'_1, m'_2|jm\rangle
\] (80)
Multiplying both sides by \( \langle m_1, m_2 | \), we have,
\[
\sqrt{(j \mp m)(j \pm m + 1)} \langle m_1 m_2 | j, m \pm 1 \rangle
= \sqrt{(j_1 \pm m_1)(j_1 \mp m_1 + 1)} \langle m_1 \mp 1, m_2 | jm \rangle
+ \sqrt{(j_2 \pm m_2)(j_2 \mp m_2 + 1)} \langle m_1, m_2 \mp 1 | jm \rangle
\]
which are very useful in deriving new C-G coefficients.

Let us consider a well-known example: spin and orbit coupling of the electron in a hydrogen atom. The electron has a spin wave function \(| m_s \rangle\) (we denote it as \(\chi_{m_s}\)). It can also be in the orbital state \(| lm \rangle\) with wave function \(Y_{lm}\). We define the total angular momentum operator as
\[
\vec{J} = \vec{L} + \vec{S}
\]
The states with a definite total angular momentum \(j\) are denoted as \(| jm_j \rangle\) which are common eigenstates of \(\vec{J}^2, J_z, \vec{L}^2\) and \(\vec{S}^2\). For a given \(l\), \(j\) can take two possible values: \(l + 1/2\) and \(l - 1/2\) (except for the case of \(l = 0\) for which \(j = 1/2\))

Now let us find the C-G coefficients. One of the recursion relation for \(j = l + 1/2\) is
\[
\sqrt{(l + 1/2 + m + 1)(l + 1/2 - m)} \langle m - 1/2, 1/2 | l + 1/2, m \rangle
= \sqrt{(l + m + 1/2)(l - m + 1/2)} \langle m + 1/2, 1/2 | l + 1/2, m + 1 \rangle
\]
where the lower sign is taken and \(m_1 = m - 1/2, m_s = 1/2\).

From this one finds,
\[
\langle m - 1/2, 1/2 | l + 1/2, m \rangle = \sqrt{\frac{l + m + 1}{2l + 1}} \langle l, 1/2 | l + 1/2, 1/2 \rangle = \sqrt{\frac{l + m + 1}{2l + 1}} \]
We can determine the other CG coefficient in a similar way. Finally, the states with a good \(j\) is
\[
| j = l \pm 1/2, m \rangle = \pm \sqrt{\frac{l \pm m + 1/2}{2l + 1}} Y_{lm-1/2} \chi_+ + \sqrt{\frac{l \pm m + 1/2}{2l + 1}} Y_{lm+1/2} \chi_-
= \frac{1}{\sqrt{2l + 1}} \left( \pm \sqrt{l \mp m + 1/2} Y_{l,m-1/2} \right)
\]
This is very useful when discussing the fine structure of the hydrogen spectrum.