Lecture 31, Parity (continued), Wednesday, Nov. 23

How does a wave function changes under parity? Given a position eigenstate $|\vec{r}\rangle$, it is easy to see that

$$\pi |\vec{r}\rangle = e^{i\phi} | -\vec{r}\rangle$$

(12)

This can be shown as follows.

$$\pi \hat{r} |\vec{r}\rangle = -\hat{r} \pi |\vec{r}\rangle = \hat{r} |\vec{r}\rangle$$

(13)

Therefore, $\pi |\vec{r}\rangle$ is an eigenstate of $\hat{r}$ with eigenvalue $-\vec{r}$. As a convention, we choose the phase factor $\phi = 0$ and therefore $\pi |\vec{r}\rangle = | -\vec{r}\rangle$.

Let us consider the state $|\psi\rangle$. Under parity, it becomes $\pi |\psi\rangle$. Making a scalar product with $\langle \vec{r}|$, we find

$$\psi(\vec{r}) \rightarrow \psi(-\vec{r})$$

(14)

Therefore the argument of the wave function changes the sign under parity.

For a plane wave state $\psi_p(\vec{r}) = \exp(i\vec{r} \cdot \vec{p})$, the parity transformation yields $\exp(-i\vec{r} \cdot \vec{p}) = \psi_{-p}(\vec{r})$. Therefore the state is different unless the momentum is zero. This is because the momentum operator does not commute with parity.

For the hydrogen atom, the wave function is $\psi_{nlm} = R_{nl} Y_{lm}(\theta, \phi)$ if we ignore the spin wave function. Under parity, we have

$$Y_{lm}(\pi - \theta, \phi + \pi) = (-1)^l Y_{lm}(\theta, \phi)$$

(15)

Therefore the wave functions are eigenfunctions of the parity! This is because the angular momentum operator commutes with parity and hence they have common eigenstates. For $l = \text{even}$, the wave function is even under the parity, and for $l = \text{odd}$, the wave function is odd.

Parity Selection Rule: An atom in an excited state is not stable. It can decay to the lower states through emitting electromagnetic radiation. Since the photon wave length is much longer than the dimension of the atom, we may approximate the interactions between the electromagnetic currents and the photon fields in terms of multipole expansion. The most important is the dipole interactions which is of the electric origin. The magnetic dipole interaction is suppressed by $v/c$ because the system is non-relativistic. Then we have electromagnetic quadrupole, octepole, etc interactions.
Knowing the parity of the wave functions, we can find the selection rules for matrix elements. Consider for example, the dipole radiation of an atom. The relevant operator is $\vec{d} = e \vec{r}$ which changes the sign under parity transformation. Now consider the matrix element $\langle n' l' m'| \vec{d} | n l m \rangle$, we can write,

$$\langle n' l' m'| \vec{d} | n l m \rangle = \langle n' l' m'| \pi \dagger \pi \vec{d} \pi \dagger \pi | n l m \rangle = (-1)^{l' + 1 + l} \langle n' l' m'| \vec{d} | n l m \rangle$$

(16)

Therefore the matrix element is non-vanishing only if $l' + l + 1$ is even. The dipole transition can only happen between states with different even-oddness of the orbital angular momentum. The above selection rule also reflects that the electromagnetic interactions conserve parity.

When a hamiltonian commutes with parity operator

$$[\pi, H] = 0$$

(17)

we say the system is invariant under parity. The electromagnetic interactions are known to be invariant under parity, so are the gravitational interactions. The strong interactions which bind the protons and neutrons together to form atomic nuclei are also parity-invariant. Therefore, for a long time, physicists believed that parity must be a good symmetry for all laws of physics. However, in 1956, Lee and Yang realized there was no experiment up to that time which verified parity invariance for the weak interactions, which are responsible, among others, for the beta decay of atomic nuclei. Hence they proposed a set of experiments to test the parity symmetry. C. S. Wu and her collaborators first discovered the parity non-conservation in the decay of the polarized $^{60}$Co nucleus, and since then the parity is no longer considered as a sacred symmetry.

Consider a one-dimensional potential

$$V = \infty, \quad |x| > a + b$$
$$V = 0, \quad a + b > |x| > a$$
$$V = V_0 > 0, \quad |x| < a$$

(18)

which has two wells symmetrically situated around the origin. There is a barrier of height $V_0$ which separate the two wells. The system is clearly invariant under parity, $[H, \pi] = 0$.

The ground state wave function is a symmetric wave function $\psi_e$ with energy $E_1$. The first excited state has an antisymmetric wave function $\psi_o$ with
energy $E_2$. As we discussed before, both wave functions are eigenfunctions of parity because the Hamiltonian is symmetric.

As the barrier $V_0 \to \infty$, $E_1$ and $E_2$ becomes degenerate. This degeneracy can be understood as follows: In the infinite $V_0$ limit, we have two separated infinite wells, each having the same ground state energy. $E_1$ and $E_2$ becomes this degenerate energy.

The ground state wave functions are $\psi_e$ and $\psi_o$, or any linear combination of the two in the degenerate subspace of dimension-2. However, I am going to argue that the most natural choice of the two degenerate eigenfunctions are

$$\psi_{1,2} = \frac{1}{2}(\psi_e \pm \psi_o)$$

which are separately the ground states of the two separate wells. Clearly, they are not eigenstates of parity. In fact, under parity operation, one has,

$$\pi \psi_1 = \psi_2, \quad \pi \psi_2 = \psi_1$$

Technically, when we have two separate wells (worlds), it is easy to create a ground state in one of them. It is very difficult, however, to generate a state which has non-vanishing amplitudes both wells. The fundamental reason for this is that although we have a perfectly symmetric potential, any external perturbation in the process of creating the state will not be symmetric. Any non-symmetric effects will break the degeneracy and leads to $\psi_{1,2}$ as the eigenstates.

When the system in one of the ground states $\psi_1$ or $\psi_2$, we say that the parity is spontaneously broken because the ground state does not have parity symmetry.

There are many examples of the spontaneous symmetry breaking in nature. For example, in solid state physics, a sample of magnetic material above the critical temperature has spherical symmetry SO(3) if it is shaped like a sphere. One cannot tell the difference if one rotates the sample by any angles around the center. However, rotational symmetry is spontaneously broken below the critical temperature: The system has a net magnetization pointing in some direction. Now one can tell if the system has been rotated or not. In Standard Model physics, the electroweak interaction has a $U(1) \times SU(2)$ symmetry. However, the symmetry is spontaneously broken in the ground state. As a result, the W and Z bosons which mediate the weak interactions acquire non-vanishing masses, unlike the photon which stays massless.